Research Statement

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I. Introduction

I have found my research in topology to be compelling and entirely rewarding. It has been a driving force throughout my graduate and postdoctoral career, providing motivation for all arenas of my professional life, including instruction. Indeed, it is impossible to separate my "research" from "teaching"; these are linked in many ways into a connected whole. Conversations with students, whether in the context of a course or an independent study, stimulate my research ideas, and engaging in research helps me reflect on the process of learning mathematics, general problem solving, and understanding proof.

My work has focused in classic point-set topology with a somewhat set-theoretic angle. It has centered on two broad topics: homogeneous topological spaces and the partial order of Hausdorff topologies on a fixed set. Homogeneous spaces are spaces in which all points share the same topological properties. Many commonly studied spaces are homogeneous, such as the real line and the unit circle with their usual topologies. Another important class of homogeneous spaces is topological groups; i.e. groups with a topology that interacts well in a certain sense with the group structure. As such the study of homogeneous spaces represents a synergy of topological and algebraic tools. A striking result in this area is the following. The unit interval $[0, 1]$ is *not* homogeneous as the topology at the endpoints is different than other points. However, surprisingly, the Hilbert Cube $[0, 1]^{\aleph_0} = [0, 1] \times [0, 1] \times [0, 1] \cdots$ *is* homogeneous, as shown by Keller in 1930's. The "corner" points in this space somehow fold into the rest of the space and become topologically indistinguishable from the others. We say that the space $[0, 1]$ is *power homogeneous*. Results like these make the study of homogeneous spaces incredibly rich and fascinating, and accessible to advanced undergraduate students.

In my work I have shown that well-known cardinality bounds on Hausdorff topological spaces can be much improved if the space is known to be homogenenous. For example, it is straight-forward to show that a Hausdorff space with a countable basis has cardinality at most $\mathfrak{c}$, the cardinality of the continuum. However, in my work with co-author G.J. Ridderbos, we have shown that if the space is homogeneous then the countable basis condition is much stronger than is needed to ensure the same cardinality bound. This result was described by well-known topologist A.V. Arhangel'skii in personal communication as, "very beautiful, interesting, and definitive." I feel this theorem is the capstone result of my postdoctoral career. It has furthermore lead to a productive, long-term, and rewarding collaboration with Ridderbos. In the summer of 2008 I had the opportunity to work with Ridderbos in Amsterdam at Vrije Universiteit, where we obtained new results and surprising generalizations.

I have also focused on the equally intriguing study of the partial order of Hausdorff topologies on a set, partially ordered by inclusion. This partial order is a natural and important object of study in topology as the large majority of topologies encountered in research possess the Hausdorff property. In this partial order a "jump" may occur between two topologies. This means no other Hausdorff topology lies strictly between these two. Such a jump represents in some sense something very fundamental about the structure of the Hausdorff partial order. Knowing, for example, exactly which topologies can be on either side of this jump, so-called *lower* and *upper* topologies, gives us a strong glimpse into the nature of this partial order and Hausdorff topologies in general. In my work I have obtained a very useful characterization of an upper topology, one that has been used in subsequent papers by O. Alas, et al [2], and C. Costantini [7]. In addition I have answered a question of Alas and Wilson by constructing an example of a lower topology with certain properties. Furthermore, in post-thesis collaboration with my advisor, J. R. Porter, we have generalized results of Alas, Wilson, and Costantini, obtained new results concerning remote points, and explored the connection between the Hausdorff partial order and open ultrafilters.

Although these are seemingly disparate areas of research, in each I have focused in part on the role played by H-closed spaces. A Hausdorff topological space is *H-closed* if it is closed in every Hausdorff space in which it is embedded, or equivalently, if every open cover has a subfamily whose union is dense. H-closed is then a weakening of the compactness property. Compactness is a very useful property for space to possess, and sometimes it is really stronger than is needed. In many situations H-closed will suffice to obtain the usual benefits of compactness. Certain notions of convergence that hold in compact spaces, for example, hold in a somewhat weaker sense in H-closed spaces. Or perhaps one does not actually need a finite subcover, only a finite subfamily whose *closures* cover. In a compact Hausdorff space every closed subspace is compact. However, in an H-closed space the closure of every *open* set is still H-closed, and this may be all that is needed. These spaces have been studied by many, notably Tychonoff in the 1930's, Katětov in the 1940's, and by Dow, Porter, Stephenson, Vermeer, Woods, Alas, Wilson, and others in more recent decades.

II. Homogeneity

A topological space is *homogeneous* if for all $x, y \in X$ there exists an autohomeomorphism $h : X \to X$ such that $h(x) = y$. A space X is *power homogeneous* if some power of the space is homogeneous, i.e. X^κ is homogeneous for some cardinal κ. Much recent research in the area of homogeneity has focused on obtaining cardinality bounds for spaces with one of the above properties. Van Douwen's fundamental 1978 paper [8] introduced crucial techniques in the study of these bounds such as invariant families and clustering. He showed that if X is a Hausdorff power homogeneous space with a countable π-*base*, a weak type of base, then the cardinality of X is at most $\mathfrak{c}$. In 2007 G.J. Ridderbos and I proved, using the Erdös-Rado partition relation, that the countable π-base condition is actually much stronger than is needed to obtain the same bound. Our main result, published in [6], is as follows:

Theorem (Carlson, Ridderbos)**.** *If X is a power homogeneous Hausdorff space with the countable chain condition and countable π-character, then the cardinality of X is at most $\mathfrak{c}$.*

A space has the *countable chain condition* (c.c.c) if every family of pair-wise disjoint open sets is countable. For example, the real line is c.c.c. A space has *countable π-character* if it has a countable "local" π-base at every point. As every space with a countable π-base has countable π-character and is c.c.c, our result truly is a substantial improvement over the above van Douwen bound. Although the Erdös-Rado Theorem has been used previously to prove cardinality bounds on topological spaces, our result represents the first use of the Erdös-Rado Theorem in the context of homogeneity. A.V. Arhangel'skii has noted that this fusion of concepts may open new doors in solving long-standing problems in this area. It should be noted that before we obtained this result I had previously proved it in the case where the space is *Urysohn*, a slightly stronger separation axiom than Hausdorff. See [3].

Ridderbos and I have recently explored the question of whether every compact, homogeneous space with countable π-character has cardinality at most $\mathfrak{c}$. In other words, if the space is known to be compact, can we get rid of the c.c.c condition in the above theorem? In our work on this problem in Amsterdam in the summer of 2008 we managed to obtain a counterexample to a related question and prove a very different cardinality bound for Hausdorff homogeneous spaces. Thus the compactness question has motivated our recent homogeneous work while opening up new doors and directions. Either a positive or negative answer to this question would be a substantial contribution to the field, and we feel we are close to such an answer.

Question. *Does every compact homogeneous space with countable π-character have cardinality at most $\mathfrak{c}$?*

III. The Partial Order of Hausdorff Topologies

Given a set X we can partially order the Hausdorff topologies on X under inclusion. A minimal element in this partial order is called *minimal Hausdorff*. A Hausdorff topology, then, is minimal Hausdorff if it does not contain a strictly coarser Hausdorff topology. Equivalently, a Hausdorff space is minimal Hausdorff if and only if it is H-closed (see above) and semiregular, a weakening of the "regular" separation axiom.

In the partial order of Hausdorff topologies on a set there may be "jumps" in the following sense. Given Hausdorff topologies $\sigma \subseteq \tau$ it may be true that if μ is Hausdorff and $\sigma \subseteq \mu \subseteq \tau$ then $\mu = \sigma$ or $\mu = \tau$. In this situation there are no Hausdorff topologies strictly between σ and τ. We say σ is a *lower topology* and τ is an *upper topology.* Together they constitute a jump in the partial order.

In [1], Alas and Wilson used the notion of a maximal point to characterize lower topologies. A point p in a space X is a *maximal point* if whenever U is open and $p \in \text{cl}_X U$ then $U \cup \{p\}$ is open. They showed that a Hausdorff topology is a lower topology on a set X if and only if there exists a closed subspace $A \subseteq X$ with a maximal point. This characterization lead to important results concerning which topologies can be lower. For example, no compact topology is lower. Yet the question of characterizing upper topologies remained. Clearly a minimal Hausdorff topology cannot be an upper topology, as there is no coaser Hausdorff topology that could serve as it corresponding lower topology. But is every non minimal Hausdorff topology an upper topology? I asked this question in [4], and it was answered in the negative first under the Continuum Hypothesis by Costantini [7]. It was then answered without any set-theoretic assumptions by Alas, et al, in [2]. Central to their work is perhaps my strongest result in this area- a characterization of upper topologies in terms of *simple extension topologies*, published in [4]:

Theorem (Carlson)**.** *A topology τ is an upper topology in the partial order of Hausdorff topologies on a set X if and only if there exists a Hausdorff topology $\sigma \subsetneq \tau$ and $U \notin \sigma$ such that τ is the topology generated by $\sigma \cup \{U\}$.*

Recently, in post-thesis collaboration with my advisor Prof. Jack Porter, we have focused on the notion of an *open ultrafilter*, that is, a maximal filter of open sets. Open ultrafilters are integral to the study of maximal points, as well as so-called remote points. If Y is an extension of a space X, a point $p \in Y \backslash X$ is a *remote point* if for each nowhere dense subset $A \subseteq X$ there is an open subset containing p such that $U \cap A = \emptyset$. A fundamental 1981 result of van Douwen [9] gave conditions for which the Stone-Čech compactification of a Tychonoff space has a remote point:

Theorem (van Douwen)**.** *If X is a non-feebly compact Tychonoff space with a countable π-base, then the Stone-Čech compactification of X contains a remote point.*

Note that a space is *feebly compact* if every countable open cover has a finite subfamily whose union is dense, that is, countably H-closed. This represents a further weakening of the compactness condition. Now, Porter and I were able to prove a result related to the above result of van Douwen in the setting of regular spaces, using the notion of a regular filter. An open filter $\mathcal{F}$ is *regular* if for each $U \in \mathcal{F}$ there is some $V \in \mathcal{F}$ such that $clV \subseteq U$. In various ways, such regular filters act like remote points. For a regular space the Stone-Čech compactification may not exist, however we showed the following in [5], which we feel is an important contribution to the theory of remote points:

Theorem (Carlson, Porter)**.** *If X is a non-feebly compact space with a countable π-base, then some open ultrafilter on X is also a regular filter.*

In [1], Alas and Wilson showed that no compact Hausdorff space can have a maximal point and thus cannot be a lower topology. Costantini [7] generalized this by showing that no countably compact regular space contains a maximal point. We were able to generalize this much further by showing:

Theorem (Carlson, Porter)**.** *A maximal point in a Hausdorff space cannot have a neighborhood base of feebly compact neighborhoods.*

Consequences of this result are that no locally countably compact topology (which includes locally compact topologies) is a lower topology, and, surprisingly, that no Stone-Čech compactification of a feebly compact Tychonoff space contains a remote point.

Related to the notion of a maximal point is the idea of a *submaximal space*. These are spaces in which every dense set is open, which is a very strong (and frankly odd) property for a space to possess. Porter and I continue to investigate the following question:

Question. *Does there exist a submaximal, minimal Hausdorff space without isolated points?*

In some sense, the nature of the problems surrounding the study of the Hausdorff partial order is quite fundamental. When a question is asked such as "Is every Hausdorff topology that is not minimal Hausdorff an upper topology?", we are asking a fundamental question about the structure of this partial order. As we only assume these topologies are Hausdorff, we do not have access to much complex machinery. For this reason the proofs involve rather ingenious applications of essentially "primal" techniques. It is this unique, elementary nature of this area that strongly intrigues me and prompts me to continue further study.

References

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