

ON OPEN ULTRAFILTERS AND MAXIMAL POINTS

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ABSTRACT. In this work we expand upon the theory of open ultrafilters in the setting of regular spaces. In [9], van Douwen showed that if X is a non-feebly compact Tychonoff space with a countable π -base, then βX has a remote point. We develop a related result for the class of regular spaces which shows that in a non-feebly compact regular space with a countable π -base, there exists a free open ultrafilter on X that is also a regular filter.

Of central importance is a result of Mooney [4] that characterizes open ultrafilters as open filters that are saturated and disjoint-prime. Smirnov [7] showed that maximal completely regular filters are disjoint prime, from which it was concluded that βX is a perfect extension for a Tychonoff space X . We extend this result, and results of Skljarenko [6], by showing that under certain conditions maximal regular open filters are disjoint-prime.

Open ultrafilters are integral to the study of maximal points and lower topologies in the partial order of Hausdorff topologies on a fixed set. We show that a maximal point in a Hausdorff space cannot have a neighborhood base of feebly compact neighborhoods. One corollary is that no locally countably compact Hausdorff topology is a lower topology, which was shown previously under the additional assumption of countable tightness by Alas and Wilson [1]. Another is that a maximal point in a feebly compact space is not a regular point. This generalizes results of both Carlson [2] and Costantini [3].

1. OVERVIEW

An *open ultrafilter* on a space X is an open filter on X that is maximal in the collection of all open filters, partially ordered by inclusion. In this study we expand upon the theory of open ultrafilters by showing that certain results involving extensions of Tychonoff spaces have analogous results in the setting of regular spaces. These results involve open filters that either are, or are close to being, open ultrafilters. Recall that an open filter on a space X is *saturated* if the open filter contains every open dense subset of X . Observe that open ultrafilters are saturated. Also note that an open filter \mathcal{F} on a space X is saturated if and only if $\{U \cup X \setminus cl_X U : U \in \tau(X)\} \subseteq \mathcal{F}$. An open filter \mathcal{F} on a space X is *disjoint-prime* if whenever $U \in \tau(X)$ and $U \cup X \setminus cl_X U \in \mathcal{F}$, it follows that $U \in \mathcal{F}$ or

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$X \setminus cl_X U \in \mathcal{F}$. In 1993, Mooney [4] demonstrated that open ultrafilters can be characterized among open filters as being saturated and disjoint-prime.

If Y is an extension of a space X , and $p \in Y \setminus X$, we define $0^p = \{U \cap X : p \in U \text{ and } U \text{ is open in } Y\}$, the open neighborhood trace on X . Note that 0^p is an open filter on X . Van Douwen [9] proved an important result in the theory of extensions: if X is a non-feebly compact Tychonoff space with a countable π -base, then βX has a remote point, i.e. a point that is not in the closure of any nowhere-dense subset of X . If p is a remote point, then it is not hard to see that 0^p is saturated and, by the result of Mooney, an open ultrafilter. Thus the result of van Douwen is really a result about open ultrafilters. We generalize the notion of a remote point and investigate saturation and disjoint-primeness in the setting of regular spaces. Recall that an open filter \mathcal{F} on a space X is *regular* if for each $U \in \mathcal{F}$, there is some $V \in \mathcal{F}$ such that $cl_X V \subseteq U$. We abbreviated this notion by saying that \mathcal{F} is a *R-filter*. In Theorem 3.6, we show a result analogous to that of van Douwen: if X is a non-feebly compact regular space with a countable π -base, then there exists an open ultrafilter on X that is also a regular open filter. This is achieved through use of the absolute EX of a Hausdorff space X .

An open filter \mathcal{F} on a space X is *completely regular* if for each $U \in \mathcal{F}$, there is an open $V \in \mathcal{F}$ and a continuous real-valued function on X such that $f[V] = 0$ and $f[X \setminus U] = 1$. We abbreviate this notion by saying that \mathcal{F} is a *CR-filter*. As we will see in Theorem 2.19, if X is a Tychonoff space and p is any point in $\beta X \setminus X$, then 0^p is disjoint-prime. This follows from the fact that such neighborhood traces are maximal completely regular open filters. From this follows a result of Smirnov [7]: βX is a perfect extension for any Tychonoff space X , i.e. a certain condition holds on the boundaries of open sets. This was also shown by Skljarenko [6] using proximities. We will show in Theorem 2.12 and Corollary 2.16 that other well-known extensions of a Hausdorff space are also perfect. In the regular setting, we investigate the question of whether a maximal regular open filter is also disjoint-prime. We achieve the following partial results: a maximal regular filter on a normal space is disjoint-prime (Theorem 2.22), and a saturated regular filter on a regular space is an open ultrafilter (Theorem 3.7).

The theory of open ultrafilters is integral to the study of lower and upper topologies in the partial order $\Sigma_2(X)$ of Hausdorff topologies on a fixed set. A pair of topologies $\tau \subsetneq \sigma$ in $\Sigma_2(X)$ for which there is no topology $\mu \in \Sigma_2(X)$ satisfying $\sigma \subsetneq \mu \subsetneq \tau$ are called *lower* and *upper* topologies in $\Sigma_2(X)$, respectively. Such a pair constitutes a jump in the partial order $\Sigma_2(X)$. In 2003, Alas and Wilson [1] initiated the project of determining which topologies are lower or upper based on *internal* properties of the space. For example, clearly a compact space cannot be an upper topology in $\Sigma_2(X)$, as such spaces are minimal Hausdorff and thus have no coarser topology to serve as the corresponding lower. But neither is a compact Hausdorff space lower, as was shown in [1]. Alas and Wilson showed that lower topologies can be characterized in terms of maximal

points. A point p in a space X is a *maximal point* if p is non-isolated and if $p \in cl_X U$ for an open set U of X , then $U \cup \{p\}$ is an open set in X . It was also shown in [1] that maximal points can in turn be characterized in terms of open ultrafilters. Thus the machinery of open ultrafilters can be applied. For example, Alas and Wilson [1] give a proof that a compact Hausdorff space cannot contain a maximal point, and thus the topology cannot be a lower topology. However, viewing a maximal point of a space X as an open ultrafilter in an extension of X , such a result can be shown quite directly. (See Theorem 4.4). Moreover, a more general result (Theorem 4.5) can easily be shown using open ultrafilters: no locally compact Hausdorff topology is a lower topology. Such direct proofs testify to the importance of open ultrafilters in this setting.

Our most general result concerning maximal points is Theorem 4.6: A maximal point in a Hausdorff space cannot have a neighborhood base of feebly compact neighborhoods. This enables us to extend results of Alas and Wilson [1], Carlson [2], and Costantini [3]. A striking corollary is Corollary 4.8: no locally countably compact Hausdorff topology is a lower topology. This was shown under the additional assumption of countable tightness by Alas and Wilson [1]. This is contrasted with the existence in ZFC of a countably compact, countably tight Hausdorff lower topology, as shown by Carlson [2]. Note that Corollary 4.8 also generalizes Theorem 4.5, that no locally compact Hausdorff topology is a lower topology. Another corollary of Theorem 4.6 is Corollary 4.10, in which we show that a maximal point in a feebly compact space is not a regular point. A point p in a space (X, τ) is a *regular point* if whenever $p \in U \in \tau$ then there exists $V \in \tau$ such that $p \in V \subseteq cl V \subseteq U$. Corollary 4.10 was shown previously under the stronger condition of H-closed by Carlson [2], and under countable compactness by Costantini [3]. Our result abstracts the underlying property needed, feebly-compactness, that is implied by both the H-closed property and countable compactness. While a consequence of the Costantini result is that a countably compact, regular topology is not lower, the same result does not hold in general for feebly compact, regular topologies. We give an example of a feebly compact, lower topology that is in fact Tychonoff (Example 4.12).

All spaces are assumed to be Hausdorff.

2. PERFECT EXTENSIONS

Recall that a space X is *H-closed* if it is closed in every Hausdorff space in which it is embedded, or, equivalently, if every open cover of X has a finite subfamily whose union is dense in X . A space X is *feebly compact* if every countable open cover of X has a finite subfamily whose union is dense in X . Recall that a closed set F in a space X is *regular closed* if $F = cl_X U$ for an open set U of X . A space is feebly compact if and only if the countable intersection of a decreasing family of regular closed sets is non-empty. Clearly

both H-closed spaces and countably compact spaces are feebly compact. In the case of a Tychonoff space X , it is well known that X is feebly compact if and only if it is pseudocompact.

Let Y be an extension of a space X . A point $p \in Y \setminus X$ is a *remote point* if for each nowhere dense subset A of X , there is an open subset $p \in U \in \tau(Y)$ such that $U \cap A = \emptyset$. In 1981, van Douwen [9] showed the following.

Theorem 2.1. (van Douwen) *If X is a non-feebly compact, Tychonoff space with a countable π -base, then βX contains a remote point.*

If Y is an extension of X , for each open set $U \in \tau(X)$, define $o_Y U = oU = \{q \in Y : U \in 0^q\} = Y \setminus (cl_Y(X \setminus U)) = \cup\{V \in \tau(Y) : V \cap X \subseteq U\}$. The key in establishing van Douwen's result is a two step operation; the first step is to prove that the Stone-Cech compactification has this special boundary property:

$$\text{If } X \text{ is a Tychonoff space and } U \in \tau(X), \text{ then } bd_{\beta X} oU = cl_{\beta X} bd_X U.$$

This result was proven earlier by Skljarenko [6] using proximities and by Smirnov [7] before that. Skljarenko defined a compactification cY of a Tychonoff space Y to be perfect if cY has the above boundary property. We now extend this definition and a result by Skljarenko [6] to arbitrary extensions as follows:

Definition 2.2. An extension Y of a space X is called *perfect* if for each $U \in \tau(X)$, then $bd_Y oU = cl_Y bd_X U$.

Lemma 2.3. *Let Y be an extension of a space X and $U \in \tau(X)$. Then,*

- (a) *If $A \subseteq X$ and $U \cap A = \emptyset$, then $oU \cap cl_Y A = \emptyset$.*
- (b) *$bd_X U \subseteq cl_Y bd_X U \subseteq bd_Y oU$ and*
- (c) *$oU \cup o(X \setminus cl_X oU) \subseteq o(U \cup X \setminus cl_X U)$.*

Proof. For (a), note that $oU \cap A = \emptyset$ and that oU is an open set in Y . It follows that $oU \cap cl_Y A = \emptyset$. For (b), observe that $bd_X U \subseteq cl_X U \subseteq cl_Y oU$. By (a), $cl_Y bd_X U \cap oU = \emptyset$. It follows that $cl_Y bd_X U \subseteq cl_Y oU \setminus oU = bd_Y oU$. For (c), note that for $p \in Y \setminus X$, 0^p is an open filter. It is immediate that $oU \subseteq o(U \cup X \setminus cl_X U)$ and $o(X \setminus cl_X oU) \subseteq o(U \cup X \setminus cl_X U)$. \square

Theorem 2.4. *An extension Y of a space X is perfect if and only if for each $U \in \tau(X)$, $o(U \cup X \setminus cl_X U) = o(U) \cup o(X \setminus cl_X U)$.*

Proof. Suppose that $o(U \cup X \setminus cl_X U) \subseteq o(U) \cup o(X \setminus cl_X U)$. To show that $bd_Y oU \subseteq cl_Y bd_X U$, assume that $p \notin cl_Y bd_X U$. There is $V \in 0^p$ such that $V \cap bd_X U = \emptyset$. By Lemma 2.3(a), $oV \cap cl_Y bd_X U = \emptyset$. Thus, $V \subseteq U \cup X \setminus cl_X U$ and $p \in oV \subseteq o(U \cup X \setminus cl_X U)$. By hypothesis, $p \in o(U) \cup o(X \setminus cl_X U)$. If $p \in o(U)$, $p \notin bd_Y oU$. If $p \in o(X \setminus cl_X U)$, then $p \notin bd_Y oU$ as $o(U) \cap o(X \setminus cl_X U) = \emptyset$. Thus, $p \notin bd_Y oU$. Conversely, suppose $bd_Y oU \subseteq cl_Y bd_X U$. To show that $o(U \cup X \setminus cl_X U) \subseteq o(U) \cup o(X \setminus cl_X U)$, let $p \in o(U \cup X \setminus cl_X U)$. As

$(U \cup X \setminus cl_X U) \cap bd_X U = \emptyset$, by Lemma 2.3(a), $o(U \cup X \setminus cl_X U) \cap cl_Y bd_X U = \emptyset$. So, $p \notin bd_Y oU$ implying that $p \in oU \cup Y \setminus cl_Y oU$. Note that $o(X \setminus cl_X U) = Y \setminus cl_Y (cl_X U) = Y \setminus cl_Y U = Y \setminus cl_Y oU$. Thus, $p \in oU \cup o(X \setminus cl_X U)$. \square

Corollary 2.5. *An extension Y of a space X is perfect if and only if for each pair of disjoint, open sets $U, V \in \tau(X)$, $o(U \cup V) = oU \cup oV$.*

Definition 2.6. An open filter \mathcal{F} on a space X is called *prime* if whenever $U, V \in \tau(X)$ and $U \cup V \in \mathcal{F}$, it follows that $U \in \mathcal{F}$ or $V \in \mathcal{F}$. An open filter \mathcal{F} on a space X is *disjoint-prime* if whenever $U \in \tau(X)$ and $U \cup X \setminus cl_X U \in \mathcal{F}$, it follows that $U \in \mathcal{F}$ or $X \setminus cl_X U \in \mathcal{F}$.

Note that open neighborhood filters at a point are prime.

Corollary 2.7. *An extension Y of a space X is perfect if and only if for each $p \in Y \setminus X$, 0^p is a disjoint-prime open filter.*

Corollary 2.8. *If Y is an extension of a space X and for each $p \in Y \setminus X$, 0^p is an open ultrafilter, then Y is a perfect extension of X .*

Proof. Observe that every open ultrafilter on a space is disjoint-prime. The result then follows from Corollary 2.7. \square

Definition 2.9. Let X be a space. We define the *Katětov extension* κX (see [5]) as follows. $\kappa X = X \cup \{\mathcal{U} : \mathcal{U} \text{ is a free open ultrafilter on } X\}$. The collection $\{U : U \text{ is open in } X\} \cup \{U \cup \mathcal{U} : \mathcal{U} \in \kappa X \setminus X, U \in \mathcal{U}\}$ is a base for an H-closed topology on κX , in fact κX with this topology is the projective maximum amongst all H-closed extensions of X .

The following is Proposition 7.1(c)(5) in [5].

Proposition 2.10. *If (Y, τ) is an extension of a space X , then $\{oU : U \in \tau(X)\}$ is a base for a Hausdorff topology $\tau^\#$ on Y that is coarser than τ .*

$(Y, \tau^\#)$, also denoted by $Y^\#$, is known as a *strict extension* of X .

Definition 2.11. For a space X , the *Fomin extension* σX is defined to be the strict extension $(\kappa X)^\#$.

Note that by Corollary 7.1(j) in [5], σX is an H-closed extension.

Theorem 2.12. *For a space X , the H-closed extensions κX and σX are perfect extensions of X .*

Proof. For $p \in \kappa X \setminus X$ or $p \in \sigma X \setminus X$, the neighborhood trace 0^p is an open ultrafilter and hence disjoint-prime. The result then follows from Corollary 2.7. \square

For an open filter \mathcal{F} on a space X , let \mathcal{F}_s denote the open filter generated by $\{int_X cl_X U : U \in \mathcal{F}\}$.

Theorem 2.13. *Let \mathcal{U} be an open ultrafilter on a space X . Then \mathcal{U}_s is disjoint-prime.*

Proof. Let $U \in \tau(X)$ and $U \cup X \setminus cl_X U \in \mathcal{U}_s$. There is $V \in \mathcal{U}_s$ such that $int_X cl_X V = V \subseteq U \cup X \setminus cl_X U$. If $X \setminus cl_X U \notin \mathcal{U}_s$, as $X \setminus cl_X U = int_X cl_X (X \setminus cl_X U)$, there is some $T \in \mathcal{U}_s$ such that $T \cap (X \setminus cl_X U) = \emptyset$ and $int_X cl_X T = T$. As $T \cap V \subseteq U \cup X \setminus cl_X U$, $T \cap V \subseteq U$. By 2.2(a) in [5], $T \cap V = int_X cl_X (T \cap V)$. So, $T \cap V \in \mathcal{U}_s$. Thus, $U \in \mathcal{U}_s$. \square

Definition 2.14. Let X be a space. A set $A \subseteq X$ is *regular open* if $A = int(clA)$. The topology generated by the regular open sets of X forms a coarser topology on X , denoted by $X(s)$ denoted as the *semiregularization* of X . X is *semiregular* if $X = X(s)$.

Definition 2.15. Let X be a semiregular space. The extension $(\kappa X)(s)$, denoted by μX , is called the *Banaschewski-Fomin-Šanin extension* of X .

In general, μX is an H-closed, semiregular extension of X . Tikoo [8] has extended the definition of the Banaschewski-Fomin-Shanin extension to arbitrary spaces X . For $p \in \mu X \setminus X$, $0^p = \mathcal{U}_s$ for some free open ultrafilter \mathcal{U} on X . We have the following result.

Corollary 2.16. *The Banaschewski-Fomin-Shanin extension μX of a space X is perfect.*

Example 2.17. Let X be a non-regular, extremally disconnected space. Then $X(s)$ is a Tychonoff, extremally disconnected space and μX is also extremally disconnected and perfect. Also, $\mu X(s) = \beta(X(s))$ (see 7B in [5]).

Definition 2.18. An open filter \mathcal{F} on a space X is *completely regular* if for each $U \in \mathcal{F}$, there is an open $V \in \mathcal{F}$ and a continuous real-valued function on X such that $f[V] = 0$ and $f[X \setminus U] = 1$. We abbreviate this notion by saying that \mathcal{F} is a CR-filter.

Theorem 2.19. *A maximal CR-filter on a Tychonoff space is disjoint-prime.*

Proof. Let \mathcal{F} be a maximal CR-filter on the Tychonoff space X , $U \in \tau(X)$, and $\mathcal{F} \in o(U \cup X \setminus cl_X U)$. There is a $V \in \mathcal{F}$ and a bounded, continuous real-valued function f such that $f[V] = \{0\}$ and $f[bd_X U] = \{1\}$. Let $A = cl_X V \cap cl_X U$, and note that $A \subseteq U$ and $f[A] = \{0\}$. Consider the real-valued function g defined by $g(x) = f(x)$ if $x \in cl_X U$ and $g(x) = 1$ if $x \in X \setminus U$. The function g is also continuous and bounded, and $g[A] = 0$. The open filter base $\mathcal{G} = \{g^{-1}[[0, r]] : 0 < r < 1\}$ is a CR-filter. If \mathcal{G} meets \mathcal{F} , then by maximality of the CR-filter \mathcal{F} , $\mathcal{G} \subseteq \mathcal{F}$. As $g^{-1}[[0, .5]] \in \mathcal{F}$ and $g^{-1}[[0, .5]] \subseteq U$, it follows that $U \in \mathcal{F}$. If \mathcal{G} does not meet \mathcal{F} , there is some $0 < r < 1$ and $T \in \mathcal{F}$ such that $g^{-1}[[0, r]] \cap T = \emptyset$ and $T \subseteq V$. Thus, $A \cap T = \emptyset$ and $T \subseteq X \setminus cl_X U$. This shows that $X \setminus cl_X U \in \mathcal{F}$. \square

In the Stone-Cech compactification βX of a Tychonoff space X , the open neighborhood trace 0^p for points $p \in \beta X \setminus X$ are maximal CR-filters. That is, $\{0^p : p \in \beta X \setminus X\} = \{\mathcal{U} : \mathcal{U} \text{ is a free maximal CR-filter on } X\}$. This gives rise to the following corollary of Theorem 2.19.

Corollary 2.20. (Smirnov) *For a Tychonoff space X , the Stone-Cech compactification βX is perfect.*

Definition 2.21. An open filter \mathcal{F} on a space X is *regular* if for each $U \in \mathcal{F}$, there is some $V \in \mathcal{F}$ such that $cl_X V \subseteq U$. We abbreviated this notion by saying that \mathcal{F} is a R-filter.

The proof of Theorem 2.19 relies heavily on the Tychonoff property of the space X . This observation removes the need to talk about extensions like the Stone-Cech compactification being perfect and shifts the focus to the problem of determining when types of open filters defined on a space are disjoint-prime. The next step in this process is to prove that maximal regular filters defined on regular spaces are also disjoint-prime. In this direction, we have established the partial results of Theorems 2.22 and 3.7.

Corollary 2.22. *A maximal R-filter on a normal space is disjoint-prime.*

Proof. Let \mathcal{F} be an R-filter on a normal space and pick $U \in \mathcal{F}$. Then there exists $V \in \mathcal{F}$ such that $cl_X V \subseteq U$. As $cl_X V \cap X \setminus U = \emptyset$, there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that $f[cl_X V] = 0$ and $f[X \setminus U] = 1$. Thus, \mathcal{F} is a CR-filter. It follows that maximal R-filters on a normal space are maximal CR-filters. The result follows from Theorem 2.19. □

Question 2.23. *Is a maximal R-filter on a regular space necessarily disjoint-prime?*

Question 2.24. *Skljarenko [6] has characterized those compactifications of a Tychonoff space that are perfect in terms of being close to the Stone-Cech compactification and monotone maps. Is there a similar characterization of perfect H-closed extensions?*

Remark. A natural question is whether there are spaces with open filters that are disjoint-prime but not prime. Skljarenko [6] has shown that a compactification Y of a Tychonoff space X has the property that 0^p is prime for each $p \in Y \setminus X$ iff $Y =_X \beta X$ and X is normal. If X is a Tychonoff, non-normal space, there are points $p \in \beta X \setminus X$ such that the open filter 0^p is not prime, and by Corollary 2.20, 0^p is disjoint-prime.

3. SATURATED OPEN FILTERS

Recall that an open filter on a space is *saturated* if the open filter contains every open dense subset. It is immediate that open ultrafilters are saturated. An open filter \mathcal{F} on a space X is saturated if and only if $\{U \cup X \setminus cl_X U : U \in \tau(X)\} \subseteq \mathcal{F}$.

Any open filter \mathcal{F} on a space X can be expanded to a saturated open filter, as the open filter $\mathcal{F}_{\mathcal{D}}$ generated by $\mathcal{F} \cup \{U \cup X \setminus cl_X U : U \in \tau(X)\}$ is saturated. In the expansion from \mathcal{F} to the saturated open filter $\mathcal{F}_{\mathcal{D}}$, properties of \mathcal{F} may be lost. For example, the open filter \mathcal{F} generated by $\{(\alpha, \omega_1) : \alpha \in \omega_1\}$ on the the space ω_1 with the order topology is regular. But the expansion $\mathcal{F}_{\mathcal{D}}$ is not regular.

When we combined saturated and disjoint-prime, we obtain an open ultrafilter. We have the following slight variation of a result from Mooney [4].

Theorem 3.1. (Mooney) *An open filter on a space is an open ultrafilter if and only if the open filter is saturated and disjoint-prime.*

Corollary 3.2. *Let \mathcal{F} be a free, saturated CR-filter on a Tychonoff space.*

- (a) *If \mathcal{F} is a maximal CR-filter, then \mathcal{F} is an open ultrafilter.*
- (b) *\mathcal{F} is contained in a free, regular open ultrafilter.*

Proof. (a) follows directly from Theorems 3.1 and 2.19, and (b) follows directly from (a). \square

Let Y be an extension of a space X . A remote point $p \in Y \setminus X$ is equivalent to the open filter 0^p being saturated. Theorem 2.1 shows the existence of remote points in βX for certain Tychonoff spaces X ; that is, there are points $p \in \beta X \setminus X$ such that 0^p is remote, i.e., saturated. By Corollary 2.20, this saturated open filter 0^p is also disjoint-prime; that is, there is a free open ultrafilter on X that is also a CR-filter.

Our next step is to extend van Douwen's remote point result, Theorem 2.1, to regular spaces. The question is, are there free open ultrafilters on a regular space that are also R-filters? We answer this question in positive manner for regular, non-feebly compact spaces with countable π -bases. To do this we need information about the absolute of a space.

Let X be a space, and let $\theta X = \{\mathcal{U} : \mathcal{U} \text{ is an open ultrafilter on } X\}$. For $U \in \tau(X)$, let $OU = \{\mathcal{U} \in \theta X : U \in \mathcal{U}\}$. Then $\{OU : U \in \tau(X)\}$ is a base for a compact, extremally disconnected topology on θX . The subspace $EX = \{\mathcal{U} \in \theta X : a_X(\mathcal{U}) \neq \emptyset\}$ of θX is called the *absolute of X* . Furthermore, EX is a dense in θX , and $\theta X \cong_{EX} \beta EX$. The function $k_X : EX \rightarrow X$, defined by $k_X(\mathcal{U}) = a_X \mathcal{U}$, is called the *absolute map*. k_X is θ -continuous, perfect, irreducible, and onto. For a function $f : X \rightarrow Y$, we define $f^\#[A] = Y \setminus f[X \setminus A]$ for $A \subseteq X$ and note that $f^\#[A] = \{y \in Y : f^{-1}(y) \subseteq A\}$. See [5] for further reference.

Proposition 3.3. *Let X be a space.*

- (a) For $U \in \tau(X)$, $k[OU] = cl_X U = cl_X(k^\#[OU])$.
- (b) For $U \in \tau(X)$, $k^\#[OU] = \{x \in X : k^\leftarrow(x) \subseteq OU\} = int_X cl_X U$ and it follows that $k^\leftarrow[int_X cl_X U] \subseteq OU$.
- (c) For $U \in \tau(X)$, $cl_{EX} k^\leftarrow[U] = OU$.

Proof of (c). By (b), $k^\leftarrow[U] \subseteq OU$ implies that $cl_{EX} k^\leftarrow[U] \subseteq OU$. If $OU \setminus cl_{EX} k^\leftarrow[U] \neq \emptyset$, by (a) and (b), $\emptyset \neq k^\#[OU \setminus cl_{EX} k^\leftarrow[U]] \subseteq int_X cl_X U$. Thus, there is some $x \in U$ such that $k^\leftarrow(x) \subseteq OU \setminus cl_{EX} k^\leftarrow[U]$, a contradiction. So, $OU \subseteq cl_{EX} k^\leftarrow[U]$. \square

Proposition 3.4. *Let X be a space. For an open filter \mathcal{U} on EX , define $\mathcal{G} = \{U \in \tau(X) : OU \in \mathcal{U}\}$.*

- (a) \mathcal{G} is an open filter base on X .
- (b) If \mathcal{U} is an open ultrafilter, then so is \mathcal{G} .
- (c) If \mathcal{U} is an open ultrafilter that is CR and k is continuous, then \mathcal{G} is an R-filter.

Proof. (a) Suppose $U, V \in \mathcal{G}$. Then $OU, OV \in \mathcal{U}$ and $O(U \cap V) = OU \cap OV \in \mathcal{U}$. So, $U \cap V \in \mathcal{G}$.

(b) If V is open in X and V meets \mathcal{G} , then $V \cap U \neq \emptyset$ for all $U \in \mathcal{G}$. So, $OU \cap OV = O(U \cap V) \neq \emptyset$. As \mathcal{U} is an open ultrafilter, it follows that $OV \in \mathcal{U}$ and this implies that $V \in \mathcal{G}$.

(c) Let $U \in \mathcal{G}$. Then $OU \in \mathcal{U}$, $k^\leftarrow[U]$ is open, and $cl_{EX} k^\leftarrow[U] = OU$. As \mathcal{U} is an open ultrafilter, $k^\leftarrow[U] \in \mathcal{U}$. There is a $V \in \mathcal{U}$ such that $cl_{EX} V \subseteq k^\leftarrow[U]$. There is some $T \in \tau(X)$ such that $cl_{EX} V = OT$. Now, $T \in \mathcal{G}$. $cl_X T = k[OT] \subseteq k[k^\leftarrow[U]] = U$. \square

Remark. Let X be a space and \mathcal{V} be an open ultrafilter on X . Then $\mathcal{V} \in \theta X$ and $0^\mathcal{V}$ is an open filter on EX generated by $\{OU \cap EX : U \in \mathcal{V}\}$. In fact, $0^\mathcal{V}$ is a maximal CR-filter on EX . By Theorem 2.19, $0^\mathcal{V}$ is disjoint-prime. Note that $\mathcal{V} = \{U \in \tau(X) : OU \in 0^\mathcal{V}\}$.

Lemma 3.5. *Let X be a space.*

- (a) X is feebly compact iff EX is feebly compact.
- (b) If \mathcal{B} is π -base for EX , then $\{k^\#[U] : U \in \mathcal{B}\}$ is a π -base for $X(s)$.
- (c) If \mathcal{B} is a π -base for X , then $\{OU : U \in \mathcal{B}\}$ is a π -base for EX .
- (d) If X is semiregular, then $\pi(X) = \pi(EX)$.

Proof. The proof follows from Proposition 3.3. \square

Theorem 3.6. *Let X be a regular, non-feebly compact space with countable π -bases. Then some free open ultrafilter on X is also an R-filter.*

Proof. It follows that EX is non-feebly compact and has a countable π -base. By van Douwen's result, Theorem 2.1, βEX has a remote point p . Then 0^p is a free

maximal CR-filter on EX . By Proposition 3.4(c), there is a free open ultrafilter on X that is an R-filter. \square

Theorem 3.7. *A saturated, maximal R-filter on a regular space is an open ultrafilter.*

Proof. Let X be a regular space and \mathcal{F} be a saturated, maximal R-filter. A fixed maximal R-filter on a regular space is an open neighborhood filter of a point and open neighborhood filters of a point are prime and hence disjoint-prime. By Theorem 3.1, \mathcal{F} is an open ultrafilter. So, suppose that \mathcal{F} is a free saturated maximal regular filter. Let $Y = X \cup \{\mathcal{F}\}$ be the one-point extension of X with the topology generated by $\tau(X) \cup \{U \cup \{\mathcal{F}\} : U \in \mathcal{F}\}$; note that Y is regular. Let $Z = EX \cup \{\mathcal{U} \in \theta X \setminus EX : \mathcal{F} \subseteq \mathcal{U}\}$. Now, $EX \subseteq Z \subseteq \theta X$. Define $\bar{k} : Z \rightarrow Y$ be an extension of k_X and $\bar{k}(\mathcal{U}) = \mathcal{F}$ for $\mathcal{U} \in Z \setminus EX$.

Let $\mathcal{U} \in \bar{k}^{-1}(\mathcal{F})$. Then $0^{\mathcal{U}}$ is a maximal CR-filter on EX and $\mathcal{U} = \{U \in \tau(X) : OU \in 0^{\mathcal{U}}\}$. Let A be a closed nowhere dense subset of EX . Then $k_X[A]$ is a closed nowhere dense subset of X by 6.5(d) in [5] and there is $U \in \mathcal{F}$ such that $U \cap k_X[A] = \emptyset$. There is $V \in \mathcal{F} \subseteq \mathcal{U}$ such that $cl_X V \subseteq U$. Now $\mathcal{U} \in OV$. As $\mathcal{F} = \bar{k}(\mathcal{U}) \in \bar{k}[OV \cap Z] = cl_Y V$ and $cl_Y V \cap k_X[A] = \emptyset$, it follows that $(OV \cap Z) \cap A = \emptyset$. This shows that $0^{\mathcal{U}}$ is saturated. By Corollary 3.2(a), $0^{\mathcal{U}}$ is an open ultrafilter. By Proposition 3.4(c), $\mathcal{U} = \{U \in \tau(X) : OU \in 0^{\mathcal{U}}\}$ is a R-filter. But $\mathcal{F} \subseteq \mathcal{U}$ and \mathcal{F} is a maximal R-filter. Then $\mathcal{F} = \mathcal{U}$. This shows that \mathcal{F} is an open ultrafilter. \square

Corollary 3.8. *A saturated, R-filter on a regular space is contained in a regular open ultrafilter.*

4. MAXIMAL POINTS AND LOWER TOPOLOGIES

Definition 4.1. Let $\Sigma_2(X)$ denote the partial order of Hausdorff topologies on a fixed set, partially ordered by inclusion. A pair of topologies $\tau \subsetneq \sigma$ in $\Sigma_2(X)$ for which there is no topology $\mu \in \Sigma_2(X)$ satisfying $\sigma \subsetneq \mu \subsetneq \tau$ are called *lower* and *upper* topologies in $\Sigma_2(X)$, respectively. Such a pair constitutes a jump in the partial order $\Sigma_2(X)$.

Alas and Wilson [1] determined a crucial characterization of lower topologies involving so-called maximal points. A non-isolated point p in a space (X, τ) is a *maximal point* if whenever $p \in clU$, for $U \in \tau$, then $U \cup \{p\} \in \tau$.

Proposition 4.2. (Alas and Wilson) *A topology τ on a given set X is a lower topology in either $\mathcal{L}_1(X)$ or $\Sigma_2(X)$ if and only if (X, τ) has a closed subspace with a maximal point.*

Furthermore, Alas and Wilson [1] observed an important characterization of maximal points involving open ultrafilters.

Proposition 4.3. (Alas and Wilson) *A point p in a space X is a maximal point of X if and only if the trace of the open neighborhood filter at p on the subspace $X \setminus \{p\}$ is an open ultrafilter.*

A survey of results concerning maximal points reveals that it is difficult for many classes of regular spaces to contain them. Costantini [3] showed that no countably compact regular Hausdorff space can contain a maximal point. Locally compact Hausdorff spaces cannot contain maximal points, by Corollary 4.8 and Theorem 4.5 below. Corollary 4.10 shows that a maximal point in a feebly compact space cannot be a regular point. Our results on open ultrafilters show why, in some sense, it is difficult for regular spaces to contain maximal points and hence be lower. For example, if we are seeking a regular space Y with a maximal point p , then we first observe that Y is an extension of the regular subspace $X = Y \setminus \{p\}$ such that 0^p is a free regular, open ultrafilter. Indeed, by Proposition 4.3, a maximal point p in a space X can be represented as a point in the remainder of $\kappa(X \setminus \{p\})$. The goal, then, is to find a regular space with a free open ultrafilter that is also regular. In Theorem 3.6 we show that non-feebly compact, regular spaces with countable π -bases have free open ultrafilters that are also regular. Yet in general it should be viewed as difficult for a space to contain such filters, just as it is seen as difficult for βX to contain a remote point for a general Tychonoff space X . In fact, we approach this problem with care as there are regular spaces, e.g., the space ω_1 with the usual order topology, whose free open ultrafilters are never regular.

Viewing maximal points as open ultrafilters leads to nice, direct proofs of the following two theorems. Theorem 4.4 was previously shown by Alas and Wilson [1] using more involved methods. Theorem 4.5 is a new result, but in fact follows from our most general result concerning maximal points, Theorem 4.6 .

Theorem 4.4. (Alas and Wilson) *A compact Hausdorff space cannot contain a maximal point.*

Proof. Let X be compact Hausdorff with a maximal point p . Let $Y = X \setminus \{p\}$. Note that Y is Tychonoff, as Tychonoff is hereditary. As X is H-closed with a maximal point, we have $\kappa Y \approx X$. We have

$$\beta Y \leq \kappa Y \approx X \leq \beta Y,$$

where “ \leq ” is the order on the H-closed extensions of Y . The first inequality follows from the fact that κY is the projective maximum in this order. The last inequality follows from the fact that X is a compactification of Y and that βY is the projective maximum amongst compactifications of Y . (Note βY exists as Y is Tychonoff).

It follows then that $\beta Y = \kappa Y$. By problem 7B(8) in [5], it follows that Y is compact. Hence $Y = \kappa Y = X$, which is a contradiction. \square

Theorem 4.5. *A locally compact Hausdorff topology is not a lower topology.*

Proof. If X is in fact compact, then we know it cannot contain a maximal point, by Theorem 4.4. Otherwise, X has a one-point Hausdorff compactification Y in which X is open. If X has a maximal point p , then by Proposition 1.3 in [2], p is maximal in Y . But this is a contradiction since Y is compact. \square

We now give our most general result concerning maximal points that seems to be at the heart of many of the results of Alas and Wilson [1], Carlson [2] and Costantini [3].

Theorem 4.6. *A maximal point in a Hausdorff space cannot have a neighborhood base of feebly compact neighborhoods.*

Proof. Let X be Hausdorff space with a maximal point p . By way of contradiction, suppose that p has a neighborhood base of feebly compact neighborhoods. Let \mathcal{M} be a maximal pairwise disjoint family of $\{U \in \tau(X) : U \neq \emptyset, p \notin cl_X U\}$. By maximality of \mathcal{M} , $p \in cl(\bigcup \mathcal{M})$. As p is a maximal point of X , it follows that $\{p\} \cup (\bigcup \mathcal{M})$ is an open set of X . There exists an open set V and a feebly compact set F such that $p \in V \subseteq F \subseteq \{p\} \cup (\bigcup \mathcal{M})$.

Now, suppose $\mathcal{F} = \{U \in \mathcal{M} : U \cap V \neq \emptyset\}$ were finite. Then $p \in V \cap \bigcap_{U \in \mathcal{F}} (X \setminus cl U)$, an open set about p . If there exists $x \in V \cap \bigcap_{U \in \mathcal{F}} (X \setminus cl U) \cap \bigcup \mathcal{M}$, then there exists $W \in \mathcal{M}$ such that $x \in W \cap V$. Hence, $W \in \mathcal{F}$, which means $x \in X \setminus cl W$, a contradiction. Hence $V \cap \bigcap_{U \in \mathcal{F}} (X \setminus cl U)$ is an open set about p that misses $\bigcup \mathcal{M}$. This is a contradiction, since $p \in cl(\bigcup \mathcal{M})$. It follows that \mathcal{F} is infinite, and that $\{U \cap V : U \in \mathcal{F}\}$ is also infinite since \mathcal{M} is pairwise disjoint.

Let $\{T_n : n \in \omega\} \subseteq \{U \cap V : U \in \mathcal{F}\}$. As p is not isolated, either $\{p\} \cup (\bigcup \{T_{2n} : n \in \omega\}) \notin \tau(X)$ or $\{p\} \cup (\bigcup \{T_{2n+1} : n \in \omega\}) \notin \tau(X)$. Without loss of generality, suppose $\{p\} \cup (\bigcup \{T_{2n} : n \in \omega\}) \notin \tau(X)$. As p is maximal,

$$\begin{aligned} p \notin cl\left(\bigcup_{n \in \omega} T_{2n}\right) &\supseteq cl\left(\bigcup_{n \in \omega} T_{2n}\right) \cap F \\ &\supseteq cl_F\left(F \cap \bigcup_{n \in \omega} T_{2n}\right) \supseteq \bigcap_{k \in \omega} cl_F\left(\bigcup_{n \geq k} T_{2n} \cap F\right). \end{aligned}$$

Let $A = \bigcap_{k \in \omega} cl_F\left(\bigcup_{n \geq k} T_{2n} \cap F\right)$. Suppose there exists $q \neq p$ such that $q \in A$. Then $q \in F \subseteq \{p\} \cup (\bigcup \mathcal{M})$. Thus $q \in \bigcup \mathcal{M}$ and there exists $W \in \mathcal{M}$ such that $q \in W$. As $q \in A$, by letting $k = 1$ there is some m such that $W \cap T_{2m} \neq \emptyset$. Also, by letting $k = m + 1$, there exists some $l > m$ such that $W \cap T_{2l} \neq \emptyset$. This says W meets a member of \mathcal{M} that is not W , contradicting the fact that \mathcal{M} is pairwise disjoint. We conclude that $A = \emptyset$. But A is the countable intersection of the closures in F of F -open sets, contradicting that F is feebly compact. This completes the proof. \square

A space is *locally feebly compact* if every point has a neighborhood base of feebly compact neighborhoods. Likewise, a space is *locally countably compact* if every point has a neighborhood base of countably compact neighborhoods. Alas and Wilson [1] showed that a locally countably compact, countably tight space cannot contain a maximal point. Corollary 4.7 demonstrates that the locally countably compact condition can be replaced with the weaker property locally feebly compact, and that the countable tightness condition may be dropped entirely. As the property of locally countably compact is closed hereditary, in light of Proposition 4.2 Alas and Wilson concluded that no locally countably compact space of countable tightness is a lower topology in $\Sigma_2(X)$. In the case of locally feebly compact, no such conclusion can be made as the property of locally feebly compactness is not closed hereditary.

Corollary 4.7. *A locally feebly compact Hausdorff space cannot contain a maximal point.*

Corollary 4.8. *A locally countably compact Hausdorff space cannot contain a maximal point.*

In [1], Alas and Wilson gave an involved proof that a locally countably compact Hausdorff space of countable tightness is not a lower topology. The following corollary demonstrates that countable tightness is not necessary.

Corollary 4.9. *A locally countably compact Hausdorff space is not a lower topology.*

Proof. Observe that locally countably compactness is closed hereditary. \square

Note that Theorem 4.5 follows from Corollary 4.9.

Corollary 4.10. *A maximal point in a feebly compact space is not a regular point.*

Proof. A regular point in a feebly compact space has a neighborhood base of feebly compact neighborhoods, as feebly compactness is hereditary on regular-closed sets. \square

Corollary 4.11. *If X is a feebly compact, Tychonoff space then βX does not contain a remote point.*

Proof. If X is Tychonoff and feebly compact and $p \in \beta X$ is a remote point, then $X \cup p$ is still feebly compact, Tychonoff and p is a maximal point in $X \cup p$, contradicting Corollary 4.10. Thus, a Tychonoff, feebly compact space has no remote points. \square

From Corollary 4.10, it follows that countably compact regular spaces cannot contain maximal points, a result of Costantini [3], and are hence not lower as countable tightness is closed hereditary. Also by Corollary 4.10, we conclude that feebly compact regular spaces cannot contain maximal points. However, it

does not follow that no feebly compact regular topology is a lower topology, as feebly compactness is not closed hereditary. We conclude with an example of a feebly compact lower topology that is in fact Tychonoff.

Example 4.12. Let μ be an ordinal with uncountable cofinality. The ordinal $\mu + 1$ with the order topology is compact and as μ has uncountable cofinality, the subspace μ is countably compact. Let $p \in \beta\omega \setminus \omega$ and let X be defined as follows:

$$X = (\mu \times \beta\omega) \cup (\{\mu\} \times (\omega \cup \{p\})).$$

The space X is a dense subspace of the product space $(\mu + 1) \times \beta\omega$ which is compact. So, X is Tychonoff.

Since μ is countably compact and $\beta\omega$ is compact, it follows that $\mu \times \beta\omega$ is countably compact. Thus, for any countable open cover of X , there is a finite subcover of $\mu \times \beta\omega$. But $\mu \times \beta\omega$ is dense in X , so, the closure of the union of the finite subcover of $\mu \times \beta\omega$ is X . That is, X is feebly compact.

The subspace $\{\mu\} \times (\omega \cup \{p\})$ is closed in X and p is a maximal point in $\{\mu\} \times (\omega \cup \{p\})$. Hence, X is a lower topology.

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