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Non-regular power homogeneous spaces

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Abstract

We show that the cardinality of any space X with Δ-power homogeneous semiregularization that is either Urysohn or quasiregular is bounded by $2^{c(X)\pi\chi(X)}$. This improves a result of G.J. Ridderbos who showed this bound holds for Δ-power homogeneous regular spaces. By introducing the notion of a local $\pi\theta$-base, we show that this bound can be further sharpened. We also show that no H-closed extremally disconnected space is power homogeneous. This is a variation of a result of K. Kunen who showed that no compact F-space is power homogeneous.

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Keywords: Power homogeneity; Density; π-character; H-closed; Urysohn

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1. Introduction

A space X is *homogeneous* if for all $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. A space is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous. A weaker form of power homogeneity, introduced by G.J. Ridderbos in [5], is Δ-power homogeneity. X^κ is *Δ-homogeneous* if for every x, y in the diagonal of X^κ there exists a homeomorphism $h : X^\kappa \to X^\kappa$ such that $h(x) = y$. If X^κ is Δ-homogeneous for some cardinal κ, then we say that X is *Δ-power homogeneous*. For a space X, the cardinal invariants $d(X)$, $c(X)$, $w(X)$, $\pi w(X)$, $\chi(X)$, and $\pi\chi(X)$ denote the *density, cellularity, weight*, *π-weight*, *character*, and *π-character* of X, respectively. By $\mathrm{RO}(X)$ we denote the *regular open* sets of X. All spaces under discussion are Hausdorff and we make the convention that all cardinal invariants are at least $\aleph_0$. For all undefined notions see Engelking [1].

In 1978 E. van Douwen showed in [7] that $|X| \leqslant 2^{\pi w(X)}$ for any power homogeneous space X. As $|X| \leqslant 2^{w(X)}$ for any space, this fundamental result demonstrated that in the presence of homogeneity well-known cardinality bounds can be improved. Extending techniques of van Douwen, in 2005 J. van Mill [8] showed that if X is power homogeneous and compact then $|X| \leqslant w(X)^{\pi\chi(X)}$. This bound is equivalent to $d(X)^{\pi\chi(X)}$ for compact spaces. In [5] Ridderbos recently introduced new techniques and showed that $|X| \leqslant d(X)^{\pi\chi(X)}$ for any Δ-power homogeneous space. In particular, no compactness condition is required. This result sharpens the van Douwen bound for the cardinality of any Δ-power homogeneous space. Using a result of Šapirovskiĭ [6] that $d(X) \leqslant \pi\chi(X)^{c(X)}$ for any regular

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space X, Ridderbos noted in [5, Corollary 3.5] that $|X| \leqslant 2^{c(X)\pi\chi(X)}$ for any regular Δ-power homogeneous space X. By modifying the Šapirovskiĭ and Ridderbos results, we show that the property of regularity can be replaced by either Urysohn or a weaker form of regularity known as quasiregularity. Furthermore, we show only the semiregularization of X needs to be Δ-power homogeneous.

2. Non-regular spaces, homogeneity, and a result of Šapirovskiĭ

Various weaker forms of regularity are of importance in our discussion. A space X is *quasiregular* if for every non-empty open set U there exists a non-empty open set V such that $\mathrm{cl}_X V \subseteq U$. X is *semiregular* if $\mathrm{RO}(X)$ forms a basis for X. Clearly any regular space is quasiregular and semiregular, but there is no universal relationship between quasiregularity and semiregularity. For any space X, the topology having $\mathrm{RO}(X)$ as its basis forms a semiregular Hausdorff topology on the same underlying set as X. We call this topology the *semiregularization* of X, denoted by X_s. Note that the topology on X_s is coarser than the topology on X. Now, if X is homogeneous it comes as no surprise that X_s is also homogeneous, as the topology on X is modified uniformly throughout the space to obtain X_s. We make this precise in the following straightforward yet useful proposition.

Proposition 2.1. *If a space X is homogeneous, power homogeneous, or Δ-power homogeneous then X_s has the same property.*

Proof. Suppose X is homogeneous and pick $x, y \in X$. By homogeneity of X there exists a homeomorphism $h: X \to X$ such that $h(x) = y$. As h is a bijection, we need only show that $h: X_s \to X_s$ is a homeomorphism by showing it is open and continuous. Let U be open in X_s and pick $z \in h[U]$. There exists $w \in U$ such that $z = h(w)$. As the regular open sets of X form a basis in X_s, there exists $R \in \mathrm{RO}(X)$ such that $w \in R \subseteq U$. Hence $z \in h[R] \subseteq h[U]$. As $\mathrm{RO}(X)$ is invariant under homeomorphisms of X, it follows that $h[R] \in \mathrm{RO}(X)$ and hence $h[U]$ is open in X_s. Similarly, $h: X_s \to X_s$ is continuous.

Now suppose X^κ is homogeneous for some cardinal κ. Pick $x, y \in X^\kappa$. There exists a homeomorphism $h: X^\kappa \to X^\kappa$ such that $h(x) = y$. By the above, $h: (X^\kappa)_s \to (X^\kappa)_s$ is a homeomorphism. But as $(X^\kappa)_s \approx (X_s)^\kappa$ [4, Proposition 2.2(j)] it follows that $h: (X_s)^\kappa \to (X_s)^\kappa$ is a homeomorphism. This shows $(X_s)^\kappa$ is homogeneous and that X_s is power homogeneous. If X is Δ-power homogeneous, one can show X_s is Δ-power homogeneous in a similar manner. □

For many cardinal invariants, the invariant on X_s is at most the invariant on X. This is true for the weight $w(X)$, the π-weight $\pi w(X)$, and other invariants. In particular this is true for the π-character $\pi\chi(X)$ and the cellularity $c(X)$. In fact, with $c(X)$ we have equality. We state this as a lemma, the proof of which we omit.

Lemma 2.2. *For any space X, $c(X_s) = c(X)$ and $\pi\chi(X_s) \leqslant \pi\chi(X)$.*

Given a space X, the cardinal invariant $d_\theta(X)$, closely related to the density $d(X)$, is defined as follows. A subspace $D \subseteq X$ is *θ-dense* if $D \cap \mathrm{cl}_X U \neq \emptyset$ for every non-empty open set U of X. The *θ-density* $d_\theta(X)$ is the least cardinality of a θ-dense subspace of X. Observe that $d_\theta(X) \leqslant d(X)$ for any space X. We note the following, the proof of which is straightforward.

Lemma 2.3. *If X is quasiregular then $d_\theta(X) = d(X)$.*

Šapirovskiĭ showed in [6] that $d(X) \leqslant \pi\chi(X)^{c(X)}$ for regular spaces. However, if we replace $d(X)$ with $d_\theta(X)$ the regularity requirement can be dropped. In fact, it seems no separation axiom is required at all. We show this by modifying a proof of the Šapirovskiĭ result given in [2, 2.37].

Theorem 2.4. *For any space X, $d_\theta(X) \leqslant \pi\chi(X)^{c(X)}$.*

Proof. For $p \in X$ let $\mathcal{B}_p$ be a local π-base at p such that $|\mathcal{B}_p| \leqslant \pi\chi(X)$. For $A \subseteq X$, define $\mathcal{B}_A$ as $\mathcal{B}_A = \bigcup\{\mathcal{B}_p\colon p \in A\}$. We now define a map

$$G\colon [X]^{\leqslant c(X)} \to [X]^{\leqslant \pi\chi(X)^{c(X)}}.$$

First, for $A \in [X]^{\leqslant c(X)}$, define

$$\mathcal{C}_A = \Bigl\{\mathcal{U} \in [\mathcal{B}_A]^{\leqslant c(X)}\colon\ X\setminus \operatorname{cl}\bigcup\mathcal{U} \neq \emptyset\Bigr\}.$$

Then $|\mathcal{C}_A| \leqslant |\mathcal{B}_A|^{c(X)} \leqslant (|A|\cdot\pi\chi(X))^{c(X)} = \pi\chi(X)^{c(X)}$. Now for each $\mathcal{U} \in \mathcal{C}_A$ pick $p(\mathcal{U}) \in X\setminus\operatorname{cl}\bigcup\mathcal{U}$ and define $G(A) = \{p(\mathcal{U})\colon \mathcal{U} \in \mathcal{C}_A\} \in [X]^{\leqslant \pi\chi(X)^{c(X)}}$. We now apply [2, 2.24(a)] to obtain a set $A \in [X]^{\leqslant \pi\chi(X)^{c(X)}}$ such that $G(B) \subseteq A$ for all $B \in [A]^{\leqslant c(X)}$. We say that A is closed with respect to G.

We claim that A is θ-dense in X. Assume the contrary. Then there exists $\emptyset \neq U \in \tau(X)$ such that $U \subseteq \operatorname{cl} U \subseteq X\setminus A$. Now let $\mathcal{U}$ be a maximal pairwise disjoint family of members of $\mathcal{B}_A$ disjoint from $\operatorname{cl} U$. Suppose there existed $p \in A\setminus\operatorname{cl}\bigcup\mathcal{U}$. Then $p \notin \operatorname{cl} U$, hence $p \in X\setminus((\operatorname{cl}\bigcup\mathcal{U}) \cup \operatorname{cl} U)$. There exists $V \in \mathcal{B}_p \subseteq \mathcal{B}_A$ such that $V \subseteq X\setminus((\operatorname{cl}\bigcup\mathcal{U}) \cup \operatorname{cl} U)$. This contradicts the maximality of $\mathcal{U}$. Hence $A \subseteq \operatorname{cl}\bigcup\mathcal{U}$. But $|\mathcal{U}| \leqslant c(X)$ hence we can find a set $H \in [A]^{\leqslant c(X)}$ such that $\mathcal{U} \in [\mathcal{B}_H]^{\leqslant c(X)}$. Since $(\bigcup\mathcal{U}) \cap U = \emptyset$, we have $X\setminus\operatorname{cl}\bigcup\mathcal{U} \neq \emptyset$ and so $\mathcal{U} \in \mathcal{C}_H$. Consequently we have $p(\mathcal{U}) \in G(H) \subseteq A$ as A is closed with respect to G. But $p(\mathcal{U}) \in X\setminus\operatorname{cl}\bigcup\mathcal{U} \subseteq X\setminus A$, which is a contradiction. This shows A is θ-dense in X. Therefore,

$$d_\theta(X) \leqslant |A| \leqslant \pi\chi(X)^{c(X)}. \qquad \square$$

By Lemma 2.3 and Theorem 2.4 we obtain a slight improvement on the Šapirovskiĭ result.

Corollary 2.5. *If X is quasiregular then $d(X) \leqslant \pi\chi(X)^{c(X)}$.*

The *$\pi\theta$-character* of a space X, denoted by $\pi\chi_\theta(X)$ is related to $\pi\chi(X)$ in the same way $d_\theta(X)$ is related to $d(X)$. For $x \in X$, a *local $\pi\theta$-base* at x is a collection $\mathcal{B}$ of non-empty open sets such that for every open neighborhood U of x there exists $B \in \mathcal{B}$ such that $\operatorname{cl}_X B \subseteq \operatorname{cl}_X U$. The *$\pi\theta$-character* at x, denoted $\pi\chi_\theta(x, X)$, is defined as the least cardinality of a $\pi\theta$-base at x. Then $\pi\chi_\theta(X) \equiv \sup_{x\in X}\pi\chi_\theta(x, X)$. Although the $\pi\theta$-character is defined in terms of properties intrinsic to X, in fact it is equivalent to $\pi\chi(X_s)$.

Proposition 2.6. *For a space X, $\pi\chi_\theta(X) = \pi\chi(X_s)$.*

Proof. Let $\mathcal{B}$ be a $\pi\theta$-base at $x \in X$ such that $|\mathcal{B}| = \pi\chi_\theta(x, X)$. Let $\mathcal{B}' = \{\operatorname{int}_X \operatorname{cl}_X B\colon\ B \in \mathcal{B}\}$, which consists of sets open in X_s. Let $U = \operatorname{int}_X \operatorname{cl}_X U$ be an X_s-basic open set containing x. As U is also open in X, there exists $B \in \mathcal{B}$ such that $\operatorname{cl}_X B \subseteq \operatorname{cl}_X U$. Then $\operatorname{int}_X \operatorname{cl}_X B \subseteq \operatorname{int}_X \operatorname{cl}_X U = U$. This shows $\mathcal{B}'$ is a π-base at x in X_s and hence $\pi\chi(x, X_s) \leqslant |\mathcal{B}'| \leqslant |\mathcal{B}| = \pi\chi_\theta(x, X)$. Now let $\mathcal{B}$ be a π-base at x in X_s such that $|\mathcal{B}| = \pi\chi(x, X_s)$. Let U be an open set in X containing x. As $\operatorname{int}_X \operatorname{cl}_X U$ is an X_s-open set containing x there exists $B \in \mathcal{B}$ such that $B \subseteq \operatorname{int}_X \operatorname{cl}_X U$. Hence $\operatorname{cl}_X B \subseteq \operatorname{cl}_X \operatorname{int}_X \operatorname{cl}_X U = \operatorname{cl}_X U$ which shows $\mathcal{B}$ is a $\pi\theta$-base at x in X and that $\pi\chi_\theta(x, X) \leqslant |\mathcal{B}| = \pi\chi(x, X_s)$. It follows that $\pi\chi_\theta(x, X) = \pi\chi(x, X_s)$ and $\pi\chi_\theta(X) = \pi\chi(X_s)$. $\square$

Corollary 2.7. *For a space X, $\pi\chi_\theta(X) \leqslant \pi\chi(X)$. If X is semiregular then equality holds.*

Proof. This follows from Lemma 2.2, Proposition 2.6, and the fact that $X = X_s$ if and only if X is semiregular. $\square$

3. A variation on a result of Kunen

We show that Proposition 2.1 can be used to prove a modification of a 1990 result of Kunen. Recall that an *F-space* is a Tychonoff space in which every cozero set is C^*-embedded. In [3] Kunen showed that no compact F-space is power homogeneous. A property closely related to compactness is H-closed. A space X is *H-closed* if it is closed in every Hausdorff space in which it is embedded, or equivalently, if every open cover of X has a subfamily whose closures cover X. The following proposition relates the H-closed property to compactness. See Corollaries 4.8(c), 4.8(k), and Proposition 4.8(h)(8) in [4] for proofs.

Proposition 3.1. *Let X be a space.*

(a) *X is H-closed and regular if and only if it is compact.*
(b) *X is H-closed, Urysohn, and semiregular if and only if it is compact.*
(c) *X is H-closed if and only if its semiregularization X_s is H-closed.*

In view of (a) above, the H-closed property is only meaningful in the absense of regularity. Recall that a space is *extremally disconnected* if the closure of every open set is open. We note the following proposition, the proof of which is straightforward.

Proposition 3.2.

(a) *Every extremally disconnected Tychonoff space is an F-space.*
(b) *A semiregular extremally disconnected space is regular.*
(c) *If X is extremally disconnected then X_s is extremally disconnected.*

We are ready prove the following, which moves Kunen's result from compactness to H-closed in the class of extremally disconnected spaces.

Theorem 3.3. *No H-closed extremally disconnected space is power homogeneous.*

Proof. Suppose X is an H-closed extremally disconnected power homogeneous space. Then X_s is semiregular and H-closed by Proposition 3.1(c). Moreover, by Propositions 3.2(c) and 2.1 it is also extremally disconnected and power homogeneous. As X_s is extremally disconnected and semiregular it follows by Proposition 3.2(b) that X_s is regular. By Proposition 3.1(a) it follows that X_s is compact and by Proposition 3.2(a) it follows that X_s is an F-space. X_s is then a compact, power homogeneous, F-space, contradicting the result of Kunen. □

It would be natural to ask whether there exist H-closed power homogeneous F-spaces. As every F-space is Tychonoff by definition, every H-closed F-space is compact by Proposition 3.1(a). Therefore, asking whether there exist H-closed power homogeneous F-spaces is equivalent to asking whether there exist compact power homogeneous F-spaces. However, we ask

Question 3.4. Does there exist an H-closed power homogeneous space in which every cozero set is C^*-embedded?

4. Δ-power homogeneous Urysohn spaces

In [5] Ridderbos showed that the cardinality of any Δ-power homogeneous space X is bounded by $d(X)^{\pi\chi(X)}$. We aim to show that if X is also *Urysohn* then $d(X)$ can be replaced with $d_\theta(X)$ and $\pi\chi(X)$ with $\pi\chi_\theta(X)$. Furthermore, X itself need not be Δ-power homogeneous, only its semiregularization X_s. (Note that by Proposition 2.1, the class of spaces with Δ-power homogeneous semiregularization includes those that are Δ-power homogeneous.) First we give a short, elegant proof in the case where X is homogeneous.

Theorem 4.1. *If X Urysohn and X_s is homogeneous then $|X| \leqslant d_\theta(X)^{\pi\chi_\theta(X)}$.*

Proof. Suppose that X is semiregular. Then $X = X_s$ and thus X is homogeneous. Fix $p \in X$ and a local π-base $\mathcal{B}$ at p such that $|\mathcal{B}| = \pi\chi(X)$. For every $x \in X$ let h_x be an autohomeomorphism of X such that $h_x(p) = x$. Let D be a θ-dense subset of X such that $|D| = d_\theta(X)$. For every $x \in X$ and $B \in \mathcal{B}$ there exists $d(x,B) \in D \cap \mathrm{cl}_X\, h_x[B] = D \cap h_x[\mathrm{cl}_X\, B]$. Define $\Phi : X \to D^{\mathcal{B}}$ by $\phi(x)(B) = d(x,B)$. We show Φ is an injection.

Let $x \neq y \in X$ and $z = h_x^{\leftarrow}(y)$. Then $p \neq z$. As X is Urysohn there exist open sets U and V containing p and z, respectively, such that $\mathrm{cl}_X\, U \cap \mathrm{cl}_X\, V = \emptyset$. As $z \in V$, we have $y \in h_x[V]$ and so $p \in h_y^{\leftarrow} h_x[V] \cap U$. There exists $B \in \mathcal{B}$ such that $B \subseteq h_y^{\leftarrow} h_x[V] \cap U$. Then $d(y,B) \in h_y[\mathrm{cl}_X\, B] \subseteq h_x[\mathrm{cl}_X\, V]$. Since $B \subseteq U$, we have $d(x,B) \in$

$h_x[\mathrm{cl}_X B] \subseteq h_x[\mathrm{cl}_X U]$. As $h_x[\mathrm{cl}_X U] \cap h_x[\mathrm{cl}_X V] = \emptyset$, it follows that $\Phi(x)(B) = d(x, B) \neq d(y, B) = \Phi(y)(B)$. Hence $\Phi(x) \neq \Phi(y)$. This shows Φ is an injection and, by Corollary 2.7,

$$|X| \leqslant |D|^{|\mathcal{B}|} = d_\theta(X)^{\pi\chi(X)} = d_\theta(X)^{\pi\chi_\theta(X)}.$$

Suppose X is not semiregular. Then since X_s is homogeneous by assumption and Urysohn by [4, 4K(7)], we can apply the above to X_s. Noting that $\pi\chi_\theta(X_s) = \pi\chi(X_s) = \pi\chi_\theta(X)$ by Corollary 2.7 and Proposition 2.6, and that $d_\theta(X_s) = d_\theta(X)$, we have

$$|X| = |X_s| \leqslant d_\theta(X_s)^{\pi\chi_\theta(X_s)} = d_\theta(X)^{\pi\chi_\theta(X)}. \qquad \square$$

To extend Theorem 4.1 from the class of Urysohn spaces with homogeneous semiregularization to the class of space with Δ-power homogeneous semiregularization, we modify various results in [5]. Specifically, where separation of points by open sets is required in [5], we instead require that the open sets have disjoint closures. Where dense sets are used, we instead use θ-dense sets.

We adopt the notational conventions used in [5]. Fix a space X, a cardinal μ, and let $Y = X^\mu$. Let $\kappa = \pi\chi(X)$ and assume $\kappa \leqslant \mu$. Let Δ be the diagonal of Y. Fix $p \in \Delta$, a projection $\pi : Y \to X$, and a local π-base $\mathcal{U}$ at $\pi(p)$ in X such that $|\mathcal{U}| \leqslant \kappa$. For any $A \subseteq \mu$, let $\pi_A : Y \to X^A$ denote the projection and define $\mathcal{U}(A)$ as in [5]. For a point $y \in Y$, let $y_A = \pi_A(y)$. For $A \subseteq \mu$ and $C \subseteq X^A$, let $\mathrm{cl}_A C$ represent the closure of C in X^A.

In the following lemma we modify Theorem 3.1 in [5]. The proof is similar and is changed only by the use of a θ-dense set rather than a dense set.

Lemma 4.2. *Let D be a θ-dense subset of X and $h : X^\mu \to X^\mu$ a homeomorphism. If $B \in [\mu]^{\leqslant\kappa}$ then there is a set $A \in [\mu]^{\leqslant\kappa}$ such that $B \subseteq A$ and for all $U \in \mathcal{U}(A)$ there is some $e(U) \in X^\mu$ and $d(U) \in D$ satisfying*:

(1) $\pi h(e(U)) = d(U)$ *and* $e(U) \in \pi_A^{\leftarrow}[\mathrm{cl}_A U]$,
(2) $h\pi_A^{\leftarrow}(e(U)_A) \subseteq \pi^{\leftarrow}(d(U))$.

Proof. Similar to the proof of Theorem 3.1 in [5], we construct by induction an increasing sequence $\{A_n \colon n < \omega\} \subseteq [\mu]^{\leqslant\kappa}$ where $A_0 = B$ such that for all $U \in \mathcal{U}(A_n)$ there is some $e(U) \in Y$ and $d(U) \in D$ satisfying

(1) $\pi h(e(U)) = d(U)$ and $e(U) \in \pi_{A_n}^{\leftarrow}[\mathrm{cl}_{A_n} U]$,
(2) $h\pi_{A_{n+1}}^{\leftarrow}(e(U)_{A_{n+1}}) \subseteq \pi^{\leftarrow}(d(U))$.

In the Ridderbos proof, D is dense and therefore $D \cap \pi h\pi_{A_n}^{\leftarrow}[U] \neq \emptyset$ for $U \in \mathcal{U}(A_n)$. For the present proof we require that $D \cap \pi h\pi_{A_n}^{\leftarrow}[\mathrm{cl}_{A_n} U] \neq \emptyset$ for the θ-dense set D. To show this, let B be a basic open set in Y such that $B \subseteq h\pi_{A_n}^{\leftarrow}[U]$. Now, the projection π is not necessarily a closed map. However, as B is basic open, it follows that $\mathrm{cl}_X \pi[B] = \pi[\mathrm{cl}_Y B]$. Since D is θ-dense in X, it follows that

$$\begin{aligned}\emptyset &\neq D \cap \mathrm{cl}_X \pi[B] = D \cap \pi[\mathrm{cl}_Y B] \\ &\subseteq D \cap \pi\big[\mathrm{cl}_Y\big[h\pi_{A_n}^{\leftarrow}[U]\big]\big] = D \cap \pi h\pi_{A_n}^{\leftarrow}[\mathrm{cl}_{A_n} U].\end{aligned}$$

The proof continues now exactly as in Theorem 3.1 in [5]. $\square$

This leads to the following modified version of Theorem 3.4 in [5].

Theorem 4.3. *Suppose X is Urysohn and X_s is Δ-power homogeneous. Then $|X| \leqslant d_\theta(X)^{\pi\chi_\theta(X)}$.*

Proof. We begin by supposing X is semiregular. Then $X = X_s$ and hence X is Δ-power homogeneous. We proceed as in Theorem 3.4 in [5], except that D is θ-dense. The map $\Phi : \Delta \to D^{\mathcal{U}(\kappa)}$ is defined similarly. To show that Φ is an injection we choose $q, r \in \Delta$ where $q \neq r$. Using the Urysohn property we obtain open sets V_q and V_r containing $\pi(q)$ and $\pi(r)$, respectively, such that $\mathrm{cl}_X V_q \cap \mathrm{cl}_X V_r = \emptyset$. Then $p \in h_q^{\leftarrow}\pi^{\leftarrow}[V_q] \cap h_r^{\leftarrow}\pi^{\leftarrow}[V_r]$. Let B be a basic open set in Y such that $p \in B \subseteq h_q^{\leftarrow}\pi^{\leftarrow}[V_q] \cap h_r^{\leftarrow}\pi^{\leftarrow}[V_r]$. Thus $p_\kappa \in \pi_\kappa[B]$, and since $\mathcal{U}(\kappa)$ is a local π-base at p_κ there exists $U \in \mathcal{U}(\kappa)$ such that $U \subseteq \pi_\kappa[B]$. Hence

$$\begin{aligned}\mathrm{cl}_\kappa U &\subseteq \mathrm{cl}_\kappa \pi_\kappa[B] = \pi_\kappa[\mathrm{cl}_Y B] \subseteq \pi_\kappa \mathrm{cl}_Y\big(h_q^{\leftarrow}\pi^{\leftarrow}[V_q] \cap h_r^{\leftarrow}\pi^{\leftarrow}[V_r]\big)\\ &\subseteq \pi_\kappa \mathrm{cl}_Y h_q^{\leftarrow}\pi^{\leftarrow}[V_q] \cap \pi_\kappa \mathrm{cl}_Y h_r^{\leftarrow}\pi^{\leftarrow}[V_r] = \pi_\kappa h_q^{\leftarrow}\pi^{\leftarrow}[\mathrm{cl}_X V_q] \cap \pi_\kappa h_r^{\leftarrow}\pi^{\leftarrow}[\mathrm{cl}_X V_r].\end{aligned}$$

Following the Ridderbos proof and using closures where necessary, it follows from the above that $\Phi(q)(U) \in \mathrm{cl}_X V_q$ and $\Phi(r)(U) \in \mathrm{cl}_X V_r$. Thus $\Phi(q)(U) \neq \Phi(r)(U)$ and $\Phi(q) \neq \Phi(r)$. This shows Φ is injective and therefore $|X| = |\Delta| \leqslant |D|^{|\mathcal{U}(\kappa)|} = d_\theta(X)^{\pi\chi(X)} = d_\theta(X)^{\pi\chi_\theta(X)}$. This latter equation uses Corollary 2.7.

If X is not semiregular, we proceed as in the non-semiregular case of Theorem 4.1 and obtain

$$|X| \leqslant d_\theta(X)^{\pi\chi_\theta(X)}. \qquad \square$$

The following is an example of a space in which Theorem 4.3 gives an improvement over the bound given in [5, Theorem 3.4].

Example 4.4. Let Y be the unit circle with its usual topology, and let X be the unit circle with the following collection as its basic open sets:

$$\big\{U \backslash C\colon\ U \text{ is open in } Y \text{ and } C \in [X]^{<\mathfrak{c}}\big\}.$$

Observe that X is homogeneous and that $Y = X_s$. As Y is Urysohn by [4, 4K(7)] it follows that X is Urysohn. Yet X is not regular. Since $\mathrm{cl}_X(U\backslash C) = \mathrm{cl}_Y(U)$, the rationals (projected naturally onto the unit circle) are θ-dense in X. Hence $d_\theta(X) = \omega$. Also, $\pi\chi_\theta(X) = \pi\chi(X_s) = \pi\chi(Y) = \omega$. However, since the irrationals (naturally projected) are dense in X and clearly C is not dense in X for any $C \in [X]^{<\mathfrak{c}}$, it follows that $d(X) = \mathfrak{c}$. Finally, observe that $\pi\chi(X) = \mathfrak{c}$. Hence, the Ridderbos cardinality bound [5, Theorem 3.4] is

$$d(X)^{\pi\chi(X)} = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$$

while the bound given by Theorem 4.3 is

$$d_\theta(X)^{\pi\chi_\theta(X)} = \omega^\omega = \mathfrak{c}.$$

We also give a straightforward example of a nonhomogeneous space with homogeneous semiregularization.

Example 4.5. Let Y be the unit circle with its usual topology, τ. Let $\mathbb{Q}$ represent the rationals naturally projected onto Y. Let X be the unit circle with topology generated by $\tau \cup \{\mathbb{Q}\}$. It is easy to see that $X_s = Y$ and thus X_s is homogeneous. Yet X is not homogeneous as the irrationals do not have countable neighborhoods while $\mathbb{Q}$ is a countable neighborhood of every rational.

Corollary 4.6. *Suppose X is Urysohn or quasiregular. If X_s is Δ-power homogeneous then $|X| \leqslant 2^{c(X)\pi\chi_\theta(X)}$.*

Proof. If X is Urysohn then by [4, 4K(7)] X_s is Urysohn. Hence by Theorems 4.3 and 2.4, Proposition 2.6, Lemma 2.2, and Corollary 2.7 we have

$$\begin{aligned}|X| = |X_s| &\leqslant d_\theta(X_s)^{\pi\chi_\theta(X_s)} \leqslant \big(\pi\chi(X_s)^{c(X_s)}\big)^{\pi\chi_\theta(X_s)}\\ &\leqslant \big(\pi\chi_\theta(X)^{c(X)}\big)^{\pi\chi_\theta(X)} = 2^{c(X)\pi\chi_\theta(X)}.\end{aligned}$$

If X is quasiregular then it is easy to see that X_s is also quasiregular. By [5, Theorem 3.4], Corollaries 2.5 and 2.7, and Lemma 2.2 we have

$$|X| = |X_s| \leqslant d(X_s)^{\pi\chi(X_s)} \leqslant \big(\pi\chi(X_s)^{c(X_s)}\big)^{\pi\chi(X_s)} \leqslant 2^{c(X)\pi\chi_\theta(X)}. \qquad \square$$

In light of Theorem 4.3 and Corollary 4.6, we ask

Question 4.7. Is the cardinality of any Hausdorff space X with Δ-power homogeneous semiregularization bounded by $d_\theta(X)^{\pi\chi_\theta(X)}$?

Question 4.8. Is the cardinality of any Hausdorff space X with Δ-power homogeneous semiregularization bounded by $2^{c(X)\pi\chi_\theta(X)}$?

Note that by Theorem 2.4 and the inequalities in the proof of Corollary 4.7, an affirmative answer to Question 4.7 would lead to an affirmative answer to Question 4.8. Moreover, we make the following observation:

Observation 4.9. The following are equivalent:

(1) The cardinality of any Δ-power homogeneous Hausdorff space X is bounded by $2^{c(X)\pi\chi(X)}$.
(2) The cardinality of any Hausdorff space X with Δ-power homogeneous semiregularization is bounded by $2^{c(X)\pi\chi(X)}$.
(3) The cardinality of any Hausdorff space X with Δ-power homogeneous semiregularization is bounded by $2^{c(X)\pi\chi_\theta(X)}$.

Proof. (2) → (1), this is just Proposition 2.1. For (1) → (2), suppose X has Δ-power homogeneous semiregularization X_s. Then,

$$|X| = |X_s| \leqslant 2^{c(X_s)\pi\chi(X_s)} \leqslant 2^{c(X)\pi\chi(X)}.$$

The first equality above is trivial, the first inequality follows from (1) and the fact that X_s is Hausdorff. The second inequality follows from Lemma 2.2.

(3) → (2) follows from Corollary 2.7. For (2) → (3) observe that

$$|X| = |X_s| \leqslant 2^{c(X_s)\pi\chi(X_s)} = 2^{c(X)\pi\chi_\theta(X)}.$$

This follows from (1), Lemma 2.2, and Proposition 2.6. □

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