Lower and Upper Topologies in the Hausdorff Partial Order on a Fixed Set

Nathan Carlson

University of Arizona

Introduction.

Fix an arbitrary infinite set X. Denote by $\Sigma_2(X)$ the collection of Hausdorff topologies on X partially ordered by inclusion.

Definition

A Hausdorff topology on X is **minimal Hausdorff** if it contains no strictly coarser Hausdorff topology on X.

Definition

A Hausdorff space Y is **H-closed** if it is closed in every Hausdorff space in which it is embedded.

Definition

A space is **semiregular** if the regular-open sets form a basis for the space. The topology generated by the regular-open sets of any Hausdorff space (Y, τ) forms a semiregular topology on Y, denoted by Y_s. We call Y_s the **semiregularization** of (Y, τ).

Proposition

(a) A space is *H*-closed, Urysohn, and semiregular iff it is compact and Hausdorff.
(b) If Y is *H*-closed and U is open in Y then cl$U$ is *H*-closed.

Proposition

A Hausdorff topology is minimal Hausdorff if and only if it is *H*-closed and semiregular.

Definition

Suppose there exist topologies $\tau \subsetneq \sigma$ in $\Sigma_2(X)$ such that if $\tau \subseteq \mu \subseteq \sigma$ for a topology μ then $\tau = \mu$ or $\mu = \sigma$. Then τ is a **lower topology** and σ is an **upper topology** in $\Sigma_2(X)$. We say $\tau^+ = \sigma$ and $\sigma^- = \tau$.

Definition

A point p is a **maximal point** of a space with topology τ if p is not isolated and whenever $p \in \mathrm{cl}_\tau U$ for $U \in \tau$ then $U \cup \{p\} \in \tau$.

Proposition (Alas, Wilson)

A point $p \in X$ is a maximal point of (X, τ) if and only if the trace of the open neighborhood filter at p on the subspace $X \setminus \{p\}$ is an open ultrafilter.

Proposition

If p is a maximal point in an open set U of (X, τ) then p is a maximal point of (X, τ).

Theorem (Alas, Wilson)

τ is a lower topology in $\Sigma_2(X)$ if and only if (X, τ) contains a closed subspace C with a maximal point p. Moreover, the upper topology is given by $\tau^+ = \langle \tau \cup \{(X \setminus C) \cup \{p\}\}\rangle$.

Example

Consider $\kappa\omega$, the Katětov H-closed extension of ω with topology τ. For any $p \in \kappa\omega \backslash \omega$, the trace of the $(\{p\} \cup \omega)$-neighborhood filter at p on ω is the ultrafilter p. Thus p is a maximal point of the open subspace $\{p\} \cup \omega$, p is a maximal point of $\kappa\omega$, and τ is a lower topology.

Consider $p \neq q \in \kappa\omega \backslash \omega$. The associated upper topologies have the form $\tau_1^+ = \langle \tau \cup \{p\} \rangle$ and $\tau_2^+ = \langle \tau \cup \{q\} \rangle$. However, q is also a maximal point of $(\kappa\omega, \tau_1^+)$. This makes τ_1^+ lower as well as upper. Likewise, τ_2^+ is both lower and upper. The topology $\sigma = \langle \tau \cup \{p, q\} \rangle$ is the upper topology associated to the lower topologies τ_1^+ and τ_2^+. Hence, τ is a lower topology with two distinct uppers, τ_1^+ and τ_2^+ are both lower and upper, and σ is an upper topology with two distinct lowers.

Which topologies on X are lower in $\Sigma_2(X)$?

Theorem (Alas, Wilson)

The Hausdorff topology τ on X is not lower in $\Sigma_2(X)$ if any of the following hold:
(a) τ is compact.
(b) τ is locally countably compact and countably tight.
(c) τ is sequential.

Lines of investigation:

(1) Observe that the topology on $\kappa\omega$ is an example of an H-closed lower topology. What can be said in general about maximal points in H-closed spaces?
(2) Note every locally countably compact regular space is countably compact. Alas and Wilson ask: Can a countably compact, countably tight Hausdorff topology be lower?

Maximal points in H-closed spaces

Definition

A point p in a space is a **regular point** if for every open set U containing p there exists an open set V such that $p \in V \subseteq \mathrm{cl} V \subseteq U$.

Lemma (Porter, Woods)

Let p be a regular point of a crowded H-closed space Y. Then p is an accumulation point of some nowhere dense subset of Y.

Lemma

A maximal point cannot be the accumulation point of a nowhere dense set.

Theorem (C.)

A maximal point in an H-closed space is not a regular point.

A countably compact, countably tight, lower topology

Arhangel'skii defined the following in 1971:

Definition

Let (Y,τ) be a space and κ a cardinal. Define a new topology σ on Y by declaring the closure of any set $A \subseteq Y$ to be as follows:

$$\mathrm{cl}_\sigma A = \bigcup_{B \in [A]^{\leq\kappa}} \mathrm{cl}_\tau B.$$

Denote the space (Y,σ) by Y^τ_κ, or by Y_κ when τ is understood. We call Y^τ_κ the **κ-tightness modification of** (Y,τ).

Proposition

Let (Y,τ) be a space and κ a cardinal. Let σ be the topology on Y_κ. Then

$$\sigma = \{U \subseteq Y : \text{ for all } A \in [Y]^{\leq\kappa} \text{ there exists } V \in \tau \text{ such that } U \cap A = V \cap A\}.$$

Lemma

Let (Y, τ) be a space, κ a cardinal, and denote by σ the topology on Y_κ. If $A \in [Y]^{\leq\kappa}$ then $\mathrm{cl}_\tau A = \mathrm{cl}_\sigma A$.

Lemma

Let (Y, τ) be a space and suppose $d(Y) = \kappa$ for a cardinal κ. Let σ be the topology on Y_κ and let σ_s denote the topology on the semiregularization of Y_κ. Then $\sigma_s \subseteq \tau \subseteq \sigma$. In addition, if (X, τ) is H-closed, then (Y, σ) is H-closed and if (Y, τ) is minimal Hausdorff then $\sigma_s = \tau$.

Theorem (C.)

If (X, τ) is a countably compact, H-closed, separable space then X_ω is an H-closed, countably compact, countably tight, separable space.

Theorem (C.)

There exists a countably compact H-closed lower topology of countable tightness.

Proof.

Let $Y = (\beta\omega)_\omega$, the countable tightness modification of $\beta\omega$. Y is H-closed, countably compact and countably tight. It remains to show that Y is a lower topology. We do this by showing any weak P-point in ω^* is a maximal point in Y.

Let p be a weak P-point in ω^*. As p is not in the ω^*-closure of any countable set of ω^* that does not contain p, we see that p becomes isolated in ω^* as a subspace of Y. Hence $\{p\} \cup \omega$ is open in Y. Now, it is not hard to see that the trace on ω of the Y neighborhood filter at p is the ultrafilter p. This shows p is a maximal point of $\{p\} \cup \omega$ in Y. As $\{p\} \cup \omega$ is open in Y we see that p is a maximal point of Y and that Y is a lower topology. $\square$

Upper topologies

Let $\mathcal{L}_1(X)$ denote the lattice of T_1 topologies on X.

Theorem (Alas, Wilson)

(a) A T_1 topology is an upper topology in $\mathcal{L}_1(X)$ if and only if it is not the cofinite topology.
(b) A Hausdorff topology on X that is not H-closed is an upper topology in $\Sigma_2(X)$.

Theorem (C.)

A Hausdorff topology σ is an upper topology in $\Sigma_2(X)$ if and only if there exists a Hausdorff topology μ on X such that $\mu \subsetneq \sigma$ and $\sigma = \langle \mu \cup \{U\} \rangle$ for some $U \in \sigma \backslash \mu$.

Recent work

Question (C.)

Is a topology τ in $\Sigma_2(X)$ non-upper if and only if it is minimal Hausdorff?

Very recently, this question appears to have been answered:
C. Costantini: NO, assuming CH, pre-print.
R.G.Wilson and others: NO, in ZFC, paper in progress.

Recent work has been done regarding jumps in the partial orders $\Sigma_t(X)$, $\Sigma_3(X)$, $\Sigma_{lc}(X)$ by Alas, Hernandez, Sanchis, Tkachenko, and Wilson, and separately by Costantini.