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Lower and upper topologies in the Hausdorff partial order on a fixed set

Nathan Carlson

Department of Mathematics, University of Arizona, 617 N. Santa Rita Ave., PO Box 210089, Tucson, AZ 85721-0089, USA

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Abstract

In the partial order of Hausdorff topologies on a fixed infinite set there may exist topologies $\tau \subsetneq \sigma$ in which there is no Hausdorff topology μ satisfying $\sigma \subsetneq \mu \subsetneq \tau$. τ and σ are *lower* and *upper* topologies in this partial order, respectively. Alas and Wilson showed that a compact Hausdorff space cannot contain a maximal point and therefore its topology is not lower. We generalize this result by showing that a maximal point in an H-closed space is not a regular point. Furthermore, we construct in ZFC an example of a countably compact, countably tight lower topology, answering a question of Alas and Wilson. Finally, we characterize topologies that are upper in this partial order as simple extension topologies.

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1. Introduction

Fix a set X and let TOP_1 denote the lattice of T_1 topologies on X partially ordered by inclusion. Observe that TOP_1 is indeed a lattice as the discrete and cofinite topologies on X are maximum and minimum in TOP_1, respectively. That is, the discrete topology contains all T_1 topologies on X while the cofinite topology is contained in all such topologies. Let TOP_2 denote the partial order of Hausdorff topologies on X. TOP_2 may not be a lattice as infimums may not exist. The discrete topology clearly contains all topologies in TOP_2 but there may be no Hausdorff topology contained in all other Hausdorff topologies on X. Topologies that are minimal in TOP_2 are called *minimal Hausdorff*. Thus a Hausdorff topology is minimal Hausdorff if and only if it contains no strictly coarser Hausdorff topology.

In either TOP_1 or TOP_2 a "jump" may occur between two topologies in which no topology lies strictly in-between. In this situation there exist topologies $\tau \subsetneq \sigma$ in either TOP_1 or TOP_2 such that if $\tau \subseteq \mu \subseteq \sigma$ then $\tau = \mu$ or $\mu = \sigma$. We say τ is a *lower topology* and σ is an *upper topology*. We adopt the notation of Alas and Wilson in [1] and let $\tau^+ = \sigma$ and $\sigma^- = \tau$. This is actually an abuse of notation, as there may be more than one topology σ with this property, as Example 1.5 below will demonstrate. However, the meaning of τ^+ and σ^- will be clear from context.

E-mail address: ncarlson@math.arizona.edu (N. Carlson).
URL: http://www.math.arizona.edu/~ncarlson.

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Natural questions arise as to which topologies can be lower or upper topologies in either TOP_1 or TOP_2. In [1], Alas and Wilson were particularly interested in which topologies are lower in TOP_1. They give a powerful characterization of lower topologies using a notion of a maximal point. A point p is a *maximal point* of a space with topology τ if p is not isolated and whenever $U \in \tau$ and $p \in \mathrm{cl}_\tau U$ then $U \cup \{p\} \in \tau$.

Theorem 1.1. (Alas, Wilson) *A topology τ on the set X is a lower topology in either* TOP_1 *or* TOP_2 *if and only if (X, τ) has a closed subspace with a maximal point.*

We will use the following characterization of a maximal point, which is straightforward to verify.

Proposition 1.2. *A point $p \in (X, \tau)$ is a maximal point of (X, τ) if and only if the trace of the open neighborhood filter at p on the subspace $X \setminus \{p\}$ is an open ultrafilter.*

The following is a useful tool in showing points are maximal points.

Proposition 1.3. *If p is a maximal point in an open set U of (X, τ) then p is a maximal point of (X, τ).*

Proof. Let $p \in \mathrm{cl}_X V$ where $V \in \tau$. We need to show that $V \cup \{p\} \in \tau$. Since $p \in \mathrm{cl}_X V$, it is easily seen that $p \in \mathrm{cl}_U(V \cap U)$. As p is a maximal point of U, it follows that $\{p\} \cup (V \cap U)$ is an open set of U and hence is in τ. As $p \in \{p\} \cup (V \cap U) \subseteq V \cup \{p\}$ we see that $V \cup \{p\} \in \tau$. □

A space is *semiregular* if the regular-open sets form a basis for the space. The topology generated by the regular-open sets of any Hausdorff space (X, τ) forms a semiregular topology on X, denoted by X_s. We call X_s the *semiregularization* of (X, τ). As we will never denote a cardinal by "s", this will not be confused with the κ-tightness modification of (X, τ). Recall that a Hausdorff space Y is *H-closed* if it is closed in every Hausdorff space in which it is embedded or, equivalently, if every open cover of Y has a finite subfamily whose union is dense in Y. In the following proposition we catalogue important properties regarding H-closed spaces. Proofs can be found in [4].

Proposition 1.4.

(a) *A space is H-closed and regular if and only if it is compact and Hausdorff.*
(b) *A space is H-closed, semiregular, and Urysohn if and only if it is compact and Hausdorff.*
(c) *If (X, τ) is H-closed and $U \in \tau$, then $\mathrm{cl}_X U$ is H-closed.*
(d) *(X, τ) is H-closed if and only if X_s is H-closed.*
(e) *A space is minimal Hausdorff if and only if it is H-closed and semiregular.*

Does there exist a lower topology with two distinct upper topologies? Likewise, does there exist an upper topology with two distinct lowers? Finally, does there exist a topology that is both lower and upper? The following straightforward example demonstrates that the answers to all three of these questions is yes.

Example 1.5. Consider the space $\kappa\omega$, the Katětov H-closed extension of ω, and let p be any point in the remainder $\kappa\omega \setminus \omega$. Let $Z_p = \{p\} \cup \omega$, an open subspace of $\kappa\omega$. Now it is not hard to see that the trace of the Z_p neighborhood filter at p on ω is the ultrafilter p. By Proposition 1.2, p is a maximal point of the open subspace Z_p. By Proposition 1.3, p is a maximal point of $\kappa\omega$. Now there are $2^{\mathfrak{c}}$ points in the remainder. Thus $\kappa\omega$ has $2^{\mathfrak{c}}$ maximal points.

Now suppose (X, τ) is a space with two distinct maximal points p and q. Certainly $\kappa\omega$ more than fulfills this requirement. Then by Theorem 1.1 τ is a lower topology. There is an upper topology associated to each maximal point. By Theorem 1.1 these topologies have the form $\tau_1^+ = \langle \tau \cup \{p\} \rangle$ and $\tau_2^+ = \langle \tau \cup \{q\} \rangle$. It is clear that these are distinct topologies. However the topology at q in topology τ_1^+ is the same as in τ, hence q is a maximal point of (Y, τ_1^+). This makes τ_1^+ lower as well as upper. Likewise, τ_2^+ is both lower and upper. By Theorem 1.1, the topology $\sigma = \langle \tau \cup \{p, q\} \rangle$ is an upper topology associated to the lower topologies τ_1^+ and τ_2^+. Hence, τ is a lower topology with two distinct uppers, τ_1^+ and τ_2^+ are both lower and upper, and σ is an upper topology with two distinct lowers.

2. Maximal points in H-closed spaces

Using Theorem 1.1, Alas and Wilson show certain classes of spaces cannot be lower topologies. In [1, Theorem 2.1] it is shown that a compact Hausdorff space cannot contain a maximal point. It follows by Theorem 1.1 that a compact Hausdorff space cannot be lower in TOP_2. Can compactness be replaced by H-closed in this result? Preceding Theorem 2.7 in [1], Alas and Wilson answer this question by giving an H-closed modification of the usual topology on $\beta\omega$ that contains a maximal point. What we show here is that a maximal point in any H-closed space cannot be a regular point, generalizing the Alas and Wilson result that no compact Hausdorff space contains a maximal point. A point p in a space (X, τ) is a *regular point* if whenever $p \in U \in \tau$ there exists $V \in \tau$ such that $p \in V \subseteq \mathrm{cl}\, V \subseteq U$. In [3] Porter and Woods show the following:

Lemma 2.1. (Porter, Woods) *Let p be a regular point of a crowded H-closed space Y. Then p is an accumulation point of some nowhere dense subset of Y.*

Lemma 2.2. *Let p be a maximal point of a space Y, U a Y-open set, and $A \subseteq Y \setminus U$. If $p \in (\mathrm{cl}\, U) \setminus A$ then $p \notin \mathrm{cl}\, A$.*

Proof. Because p is a maximal point and $p \in \mathrm{cl}\, U$, $U \cup \{p\}$ is an open neighborhood of p missing A. □

Lemma 2.3. *A maximal point cannot be the accumulation point of a nowhere dense set.*

Proof. Let p be a maximal point in a space Y, and let N be nowhere dense in Y. We can assume that $p \notin N$. Now $U = Y \setminus \mathrm{cl}\, N$ is open and dense in Y. Thus $p \in \mathrm{cl}\, U$. As $U \cap N = \varnothing$, it follows by Lemma 2.2 that $p \notin \mathrm{cl}\, N$. □

Theorem 2.4. *A maximal point in an H-closed space is not a regular point.*

Proof. Let Y be an H-closed space. By way of contradiction suppose there exists $p \in Y$ that is both a maximal point of Y and a regular point. Let $D = \{y \in Y : y \text{ is isolated}\}$. Note that D is open.

Suppose that $p \notin \mathrm{cl}\, D$. Then there exists an open set U such that $p \in U$ and $U \cap D = \varnothing$. Then $\mathrm{cl}\, U \cap D = \varnothing$ as D is open. Suppose that there exists $y \in \mathrm{cl}\, U$ that is isolated in $\mathrm{cl}\, U$. Then there exists an open set V in Y such that $V \cap \mathrm{cl}\, U = \{y\}$. As $y \in \mathrm{cl}\, U$ it follows that $\varnothing \neq V \cap U \subseteq V \cap \mathrm{cl}\, U = \{y\}$ and that $\{y\} = V \cap U$. This shows that $\{y\}$ is isolated in Y. Hence $y \in D \cap \mathrm{cl}\, U = \varnothing$, a contradiction. Now, $\mathrm{cl}\, U$ is H-closed by Proposition 1.4(c). Therefore $\mathrm{cl}\, U$ is a crowded H-closed space containing p as a regular point. By Lemma 2.1, p is the accumulation point of some nowhere dense subset N of $\mathrm{cl}\, U$. As p is a maximal point of Y it follows that p is a maximal point of the closed subspace $\mathrm{cl}\, U$. But by Lemma 2.3, p cannot be an accumulation point of N. This is a contradiction.

Thus $p \in \mathrm{cl}\, D$. Note that $p \notin D$ as p is not isolated. Since $p \in \mathrm{cl}\, D$, D is open, and p is a maximal point, it follows that $D \cup \{p\}$ is open. As p is a regular point, there exists an open set V such that $p \in V \subseteq \mathrm{cl}\, V \subseteq D \cup \{p\}$. Thus $\mathrm{cl}\, V$ is a one-point H-closed extension of a discrete space with p as the point at infinity. Such spaces are easily seen to be compact. Let $\mathrm{cl}\, V = E \cup \{p\}$ where E is discrete. For infinite disjoint sets $F, H \subseteq E$, we have $p \in \mathrm{cl}\, F$ and $p \in \mathrm{cl}\, H$. As F is open, we have a contradiction by Lemma 2.2. We conclude that p cannot be both a maximal point and a regular point simultaneously. □

Corollary 2.5. (Alas, Wilson) *A compact Hausdorff space cannot contain a maximal point.*

Corollary 2.6. (Alas, Wilson) *A compact Hausdorff topology cannot be a lower topology.*

3. A countably compact, countably tight lower topology in ZFC

A space Y is *locally countably compact* if each point $p \in Y$ has a local base of countably compact neighborhoods. It is clear that a countably compact regular space is locally countably compact, but in general a countably compact space need not be locally countably compact, even those that are Hausdorff. Again using the powerful Theorem 1.1, Alas and Wilson in [1, Theorem 2.5] show the following.

Theorem 3.1. (Alas, Wilson) *A locally countably compact Hausdorff topology of countable tightness is not a lower topology.*

Now, the example before Theorem 2.7 in [1] is an example of a countably compact Hausdorff lower topology. (Note this is also cited as an example of a H-closed lower topology). But this space is not countably tight. In light of Theorem 3.1, it is asked in Question 2.6 in [1] whether a countably compact Hausdorff topology of countably tightness can be a lower topology. We will answer this question in ZFC in the affirmative by using the *countable tightness modification* of $\beta\omega$. In general, Arhangel'skii defined the following in [2].

Definition 3.2. Let (X,τ) be a space and κ a cardinal. Define a new topology σ on X by declaring the closure of any set $A \subseteq X$ to be as follows:

$$\mathrm{cl}_\sigma A = \bigcup_{B\in[A]^{\leqslant\kappa}} \mathrm{cl}_\tau B.$$

Denote the space (X,σ) by X_κ^τ, or by X_κ when τ is understood. We call X_κ^τ the *κ-tightness modification of* (X,τ).

Note that the space X_κ will necessarily have tightness at most κ and that $\tau \subseteq \sigma$. The following straightforward proposition presents the topology on X_κ in terms of open sets.

Proposition 3.3. *Let (X,τ) be a Hausdorff space and κ a cardinal. Let σ be the topology on X_κ. Then*

$$\sigma = \bigl\{U \subseteq X\colon \text{ for all } A \in [X]^{\leqslant\kappa} \text{ there exists } V \in \tau \text{ such that } U \cap A = V \cap A\bigr\}.$$

Lemma 3.4. *Let (X,τ) be a Hausdorff space, κ a cardinal, and denote by σ the topology on X_κ. If $A \in [X]^{\leqslant\kappa}$ then $\mathrm{cl}_\tau A = \mathrm{cl}_\sigma A$.*

Proof. Observe that by definition of the topology σ on X_κ,

$$\mathrm{cl}_\sigma A = \bigcup_{B\in[A]^{\leqslant\kappa}} \mathrm{cl}_\tau B = \mathrm{cl}_\tau A.$$

The last statement follows from the fact that $A \in [X]^{\leqslant\kappa}$. □

Lemma 3.5. *Let (X,τ) be a Hausdorff space and suppose $d(X) = \kappa$ for a cardinal κ. Let σ be the topology on X_κ and let σ_s denote the topology on the semiregularization of X_κ. Then $\sigma_s \subseteq \tau \subseteq \sigma$. In addition, if (X,τ) is H-closed, then (X,σ) is H-closed and if (X,τ) is minimal Hausdorff then $\sigma_s = \tau$.*

Proof. Let D be a τ-dense subset of X such that $|D| = \kappa$. By Lemma 3.4, $\mathrm{cl}_\sigma D = \mathrm{cl}_\tau D = X$ and hence D is σ-dense in X. Let U be a basic open set in (X,σ_s), i.e. $U = \mathrm{int}_\sigma \mathrm{cl}_\sigma U$. Then

$$X \setminus U = X \setminus \mathrm{int}_\sigma \mathrm{cl}_\sigma U = \mathrm{cl}_\sigma (X \setminus \mathrm{cl}_\sigma U) = \mathrm{cl}_\sigma\bigl((X \setminus \mathrm{cl}_\sigma U) \cap D\bigr).$$

The second statement above follows from the fact that $X \setminus \mathrm{int} A = \mathrm{cl}(Y \setminus A)$ for any set A in any space. The last statement follows as D is σ-dense and $X \setminus \mathrm{cl}_\sigma U$ is open in σ. Now, $|(X \setminus \mathrm{cl}_\sigma U) \cap D| \leqslant |D| = \kappa$ and hence by Lemma 3.4,

$$X \setminus U = \mathrm{cl}_\sigma\bigl((X \setminus \mathrm{cl}_\sigma U) \cap D\bigr) = \mathrm{cl}_\tau\bigl((X \setminus \mathrm{cl}_\sigma U) \cap D\bigr).$$

Therefore $U = X \setminus \mathrm{cl}_\tau((X \setminus \mathrm{cl}_\sigma U) \cap D) \in \tau$ showing that $\sigma_s \subseteq \tau$.

Now, if (X,τ) is H-closed then the courser topology σ_s is H-closed. By Proposition 1.4 it follows that (X,σ) is H-closed. If (X,τ) is additionally minimal Hausdorff then there is no Hausdorff topology strictly coarser than τ on X. Hence $\sigma_s = \tau$. □

Lemma 3.6. *If (X,τ) is a countably compact, H-closed, separable space then X_ω is an H-closed, countably compact, countably tight, separable space.*

Proof. By Lemma 3.5, X_ω is H-closed. To show that X_ω is countably compact, pick a countably infinite set $A \subseteq X$. Now, A contains an accumulation point in X as X is compact. By Lemma 3.4, this accumulation point of A is still an accumulation point in X_ω as A is countable. This shows X_ω is countably compact. X_ω is constructed by definition to have countable tightness. To see that X_ω is separable, observe that by Lemma 3.4, countable dense sets in X are still dense in X_ω. □

Theorem 3.7. *There exists a countably compact H-closed lower topology of countable tightness.*

Proof. Let $Y = (\beta\omega)_\omega$, the countable tightness modification of $\beta\omega$. By Lemma 3.6, Y is countably compact and countably tight. It remains to show that Y is a lower topology. We do this by showing any weak P-point in ω^* is a maximal point in Y.

Let p be a weak P-point in ω^*. As p is not in the ω^*-closure of any countable set of ω^* that does not contain p, we see that p becomes isolated in ω^* as a subspace of Y. Hence $\{p\} \cup \omega$ is open in Y. Now, it is not hard to see that the trace on ω of the Y neighborhood filter at p is the ultrafilter p. This shows p is a maximal point of $\{p\} \cup \omega$ in Y. As $\{p\} \cup \omega$ is open in Y, by Proposition 1.3 we see that p is a maximal point of Y. By Theorem 1.1, Y is a lower topology. Y is additionally H-closed by Lemma 3.6. □

We note that it follows from Theorem 2.4 that no weak P-point in $\beta\omega$ is a regular point of Y in the above proof. The author is grateful to Prof. William Fleissner for suggesting the space in Theorem 3.7.

4. Upper topologies in TOP_2

We turn our attention now to upper topologies in TOP_2. Observe that in TOP_2 minimal Hausdorff topologies clearly cannot be upper topologies as there is no strictly coarser Hausdorff topology that could serve as the corresponding lower topology. In TOP_1, the cofinite topology on the fixed set X serves the same role: there is no strictly coarser T_1 topology that could serve as the corresponding lower topology. However, Alas and Wilson show in [1] that in fact the cofinite topology is the only topology that is not an upper topology in TOP_1. They obtained the following characterization:

Theorem 4.1. (Alas, Wilson) *A T_1 topology σ is an upper topology in* TOP_1 *if and only if σ is not the cofinite topology.*

Here we are particularly interested in a characterization of upper topologies in TOP_2. It is observed after Proposition 1.7 in [1] that every Hausdorff topology on X that is not H-closed is an upper topology.

Proposition 4.2. (Alas, Wilson) *Let σ be a Hausdorff topology on X that is not H-closed. Then σ is an upper topology in* TOP_2.

We give the following characterization of upper topologies in TOP_2:

Theorem 4.3. *A Hausdorff topology σ is an upper topology in* TOP_2 *if and only if there exists a Hausdorff topology μ on X such that $\mu \subsetneq \sigma$ and $\sigma = \langle \mu \cup \{U\} \rangle$ for some $U \in \sigma \setminus \mu$.*

Proof. If σ is an upper topology, then by Lemma 1.2 in [1] there exists $p \in X$ and $U \in \sigma^-$ such that $U \cup \{p\} \notin \sigma^-$ and $\sigma = \langle \sigma^- \cup \{U \cup \{p\}\} \rangle$. This establishes one direction.

Now suppose there exists an Hausdorff topology μ on X such that $\sigma = \langle \mu \cup \{U\} \rangle$ for some $U \in \sigma \setminus \mu$. For any topology ρ on X and $x \in X$, let $\rho\langle x \rangle$ denote the open neighborhood filter in ρ at x. There exists $p \in X$ such that $U \in \sigma\langle p \rangle \setminus \mu\langle p \rangle$. Define a topology τ on X by $\tau|_{X\setminus\{p\}} = \sigma|_{X\setminus\{p\}}$ and $\tau\langle p \rangle = \mu\langle p \rangle$. Note that $\mu \subseteq \tau \subsetneq \sigma$, as $U \in \sigma \setminus \tau$, and $U \setminus \{p\} \in \tau$ since μ and hence τ is T_1. Define $C = X \setminus (U \setminus \{p\}) \cup \{p\}$ and note that C is τ-closed.

Let $\mathcal{F} = \{V \cap C\colon V \in \tau\langle p \rangle\}$. Observe that $\mathcal{F}$ is a τ-open filter on C and is therefore contained in some τ-open ultrafilter $\mathcal{U}$ on C. We show $\{p\} \notin \mathcal{U}$. If $\{p\} \in \mathcal{U}$ then there exists $T \in \tau$ such that $T \cap C = \{p\}$. Note that $T \subseteq U$ as $T \setminus \{p\} \subseteq X \setminus C \subseteq U \setminus \{p\}$. Since $U \setminus \{p\} \in \tau$, it follows that $U = (U \setminus \{p\}) \cup T \in \tau$, a contradiction. Hence $\{p\} \notin U$.

Define $\sigma^- = \{V \in \sigma\colon\ p \in V \Rightarrow V \cap C \in \mathcal{U}\}$, and note that $\sigma^- \subseteq \sigma$. As $p \in U \in \sigma$, $U \cap C = \{p\}$, and $\{p\} \notin \mathcal{U}$, it follows that $U \in \sigma \setminus \sigma^-$. Finally note that $\tau \subseteq \sigma^-$, since $\tau|_{X\setminus\{p\}} = \sigma|_{X\setminus\{p\}} = \sigma^-|_{X\setminus\{p\}}$ and if $p \in V \in \tau$ then $V \cap C \in \mathcal{F} \subseteq \mathcal{U}$ and so $V \in \sigma^-$.

Now, similar to the above Example, the trace of the σ^--open neighborhood filter at p on $C \setminus \{p\}$ is a σ^-- open ultrafilter on $C \setminus \{p\}$. By Theorem 1.1 this says σ^- is a lower topology. Hence, by Lemma 1.2 in [1],

$$(\sigma^-)^+ = \langle \sigma^- \cup \{X \setminus C \cup \{p\}\}\rangle = \langle \sigma^- \cup \{U\}\rangle = \langle \mu \cup \{U\}\rangle = \sigma.$$

The second to last equality follows from the facts that $\mu \subseteq \tau \subseteq \sigma^-$, $\sigma^- \subseteq \sigma$, and $U \in \sigma$. Note all topologies under discussion are Hausdorff as they all contain the Hausdorff topology μ. This shows σ is an upper topology in TOP_2. □

We remark that the topology μ in Theorem 4.3 need not be the lower topology that corresponds to the upper topology σ, indeed it need not be a lower topology at all.

Proposition 4.2 suggests that the property of H-closed is equivalent to the property "not-upper" in TOP_2. However, the following example of an H-closed upper topology demonstrates that these properties are not equivalent.

Example 4.4. Let τ be the usual topology on the unit interval $\mathbb{I}$, $C = \{1/n\colon\ n < \omega\}$, and $\sigma = \langle \tau \cup \{\mathbb{I} \setminus C\}\rangle$. We note that $(\mathbb{I}, \sigma)$ is Hausdorff, H-closed, its semi-regularization is $(\mathbb{I}, \tau)$, and $\tau \subsetneq \sigma$. By Theorem 4.3, σ is an upper topology.

Let $\mathcal{P}$ represent the property "not-upper" in TOP_2. Now, every minimal Hausdorff topology has property $\mathcal{P}$ as noted before Theorem 4.1. Proposition 4.2 shows every topology with property $\mathcal{P}$ is H-closed. Hence property $\mathcal{P}$ is "caught between" the property H-closed and the property minimal Hausdorff. However, the Example 4.4 shows that H-closed is not equivalent to $\mathcal{P}$ and hence $\mathcal{P}$ is either equivalent to minimal Hausdorff or is equivalent to some property strictly between H-closed and minimal Hausdorff. In light of this, we ask the following question.

Question 4.5. Is every topology in TOP_2 that is not an upper topology necessarily a minimal Hausdorff topology? Equivalently, is every H-closed nonsemiregular space an upper topology?

One approach to answering Question 4.5 is to examine spaces that are close to minimal Hausdorff spaces as a counterexample. If one starts with a minimal Hausdorff space (Y, τ), perhaps it is possible to find a finer topology σ on Y such that $\tau \subsetneq \sigma$ and (Y, σ) is close enough to the minimal Hausdorff space (Y, τ) such that (Y, σ) is also not an upper. The finer topologies that are closest to (Y, τ) are $\sigma = \langle \tau \cup \{U\}\rangle$ where $U \notin \tau$. By Theorem 4.3, (Y, σ) is an upper topology for all such σ.

Example 4.4 suggests that if (Y, σ) is an H-closed space with an infinite closed discrete subset C such that $\mathrm{cl}_{\sigma_s} C \setminus C \neq \varnothing$, then (Y, σ) is an upper. If this is true, then it may be true that every H-closed nonsemiregular space is an upper.

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References

[1] O. Alas, R. Wilson, Which topologies can have immediate successors in the lattice of T_1-topologies? Appl. Gen. Topol. 5 (2004) 231–242.
[2] A. Arhangel'skii, On bicompacta hereditarily satisfying Suslin's condition, tightness and free sequences, Sov. Math. Dokl. 12 (1971) 1253–1257.
[3] J. Porter, G. Woods, Accumulation points of nowhere dense sets in H-closed spaces, Proc. Amer. Math. Soc. 93 (1985) 539–542.
[4] J. Porter, G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, Berlin, 1988.