Games!

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Problem-solving Seminar, February 14, 2007

It appears that problems of the "game" variety made their appearance on the Putnam Exam in the early 1990's and have occured ever since with regular frequency. Most of these problems are so-called *Nim Games*, but some do not fit into any category. In a Nim Game, we consider two players, say Agamemnon and Brümhilde, who make moves alternately. Agamemnon moves first, but otherwise the rules are the same for both players unless mentioned otherwise. We let M represent the set of legal moves. If a player finds him/herself in a position from which no element of M can be played then that player loses the game. We consider games in which no draw is possible and exactly one player must lose. Furthermore, we assume that positions cannot repeat and that the set of positions P is finite.

Given that one player most lose, we can partition the set P into the set L of losing positions and the set W of winning positions. Thus $P = W \cup L$ and $W \cap L = \varnothing$. A player in a losing position is guaranteed to lose provided his or her opponent plays correctly. A player in a winning position can force a win no matter what his or her opponent does.

Observe that to be in a position in W is to force his/her opponent into a position in L. To be in a position in L means that the next move must be in a position in W. From every position in W, a move to a position in L must be possible. Also note that L must contain at least one final position f from which no element of M can be played. Typically, the problem is to identify the set L of losing positions. A useful strategy is as follows: Notice that the set L is characterized by being the only set of positions satisfing all of the following conditions. (Why?)

(1) if a position is not in L then a move can be made to a position in L.

(2) if a position is in L then no move can be made to a position in L.

(3) $f \in L$.

The strategy lies in identifying a set of positions with the above properties. We illustrate this strategy with some warm-up examples. Most of the problems below are adapted from *Problem-solving Strategies* by Arthur Engel or are taken from past Putnam Exams.

Example 1. (Bachet's Game) Initially there are n checkers on the table. Players Agamemnon and Brümhilde alternate removing i checkers, where $i \in M$, the set of legal moves. The winner is the one to take the last checker. Find the set of losing positions if $M = \{1, 2, 3, \ldots, k\}$, for a positive integer k.

Example 2. The set-up is the same as Bachet's game, with $M = \{2^k : k = 0, 1, 2, \ldots\}$. Find L.

Example 3. The set-up is the same as Bachet's game, with $M = \{p : p = 1$ or p is prime$\}$. Find L.

Example 4. (1997 Putnam Exam) Players $1, 2, 3, \ldots n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Problem 1. The set-up is the same as Bachet's Game with $M = \{p^k : p$ is a prime and $k \in \mathbb{Z}, k > 0\}$. Find L.

Problem 2. (1995 Putnam Exam) A game starts with four heaps of beans, containing $3, 4, 5$, and 6 beans, respectively. Agamemnon and Brümhilde move alternately. The set M of legal moves consists of taking **either**

(a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Problem 3. Initially there is a chip at a corner square of an $n \times n$ grid. Agamemnon and Brümhilde alternately move the chip one step in any direction in the grid, as long as that square has not already been visited. The loser is the player that cannot move.

(a) Who wins for even n?

(b) Who wins for odd n?

(c) Who wins if the chip starts on a square which is *next to* a corner square?

Problem 4. This is the game of Double Chess. The rules of chess are changed as follows: White (Agamemnon) and Black (Brümhilde) make alternately two legal moves. Show that there exists a strategy for White which guarantees him at least a draw. (There is no need to necessarily demonstrate exactly what this strategy is, nor to be deeply familiar with chess!)

Problem 5. (2002 Putnam Exam) This is the game of Determinant Tic-Tac-Toe. Agamemnon enters a 1 in an empty 3×3 matrix. Brümhilde replies with a 0 in an empty position. Play continues until the entire 3×3 matrix is completed with five 1's and four 0's. Brümhilde wins if the determinant of the matrix is 0 and Agamemnon wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Problem 6. There are two piles of pennies (minted in 1931 in San Francisco, incidentally). A move consists of removing any pile entirely and splitting the other into two piles. The loser is the player who cannot move any more. Who wins, depending on the initial conditions?

Problem 7. (2006 Putnam Exam) This is Bachet's Game, with $M = \{p-1 : p$ is prime$\}$. Prove that there are infinitely many n such that Brümhilde has a winning strategy.

Problem 8. Start with $n \geq 12$ successive positive integers. Agamemnon and Brümhilde alternately take one integer until only two integers a and b are left. Agamemnon wins if $\gcd(a, b) = 1$, and Brümhilde wins otherwise. Who wins, depending on initial conditions?

Problem 9. (2002 Putnam Exam) This game is played on a polyhedron with at least five faces such that exactly three edges radiate from each of its vertices. Agamemnon and Brümhilde play the following game. Each player, alternately, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that Agamemnon has a winning strategy.

Problem 10. (Wythoff's Game) There are two piles of checkers on the table. Agamemnon takes any number of checkers from one pile or the same number of checkers from each pile. Then Brümhilde does the same. The winner is the one to take the last chip. Positions are ordered pairs $[x(i), y(i)]$ of nonnegative integers. First find a recursive rule for the losing positions, and then a closed expression for these positions.

Problem 11. The set-up is the same as Bachet's Game with the set of legal moves given by $M = \{k^2 : k \in \mathbb{Z}, k > 0\}$. Determine L.

Problem 12. Agamemnon and Brümhilde, growing tired of playing games as the evening wears on, decide to play one final game to determine the Grand Champion of the evening. There are two piles of chips, the first with a chips and the second with b chips, respectively. Initially $a > b$. A move consists of taking a multiple of the other pile from a pile. The winner is the one who takes the last chip from one of the piles. Show that:

(a) If $a > 2b$, then Agamemnon has a winning strategy.

(b) For what β can Agamemnon force a win, if initially $a > \beta b$?

(This game of Euclid is due to Cole and Davie, *Math. Gaz.* LIII, 354-357, 1969.)