Non-regular Power Homogeneous Spaces

Nathan A. Carlson
University of Kansas
carlson@math.ku.edu

1. Homogeneity properties.

Let X be a Hausdorff space.

X is **homogeneous** if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

X is **power homogeneous** if there exists a cardinal κ such that X^κ is homogeneous.

X^κ is **Δ-homogeneous** if for every x, y in the diagonal of X^κ there exists a homeomorphism $h : X^\kappa \to X^\kappa$ such that $h(x) = y$.

If X^κ is Δ-homogeneous for some cardinal κ then X is **Δ-power homogeneous**.

2. Cardinality bounds, old and new.

If X is power homogeneous then $|X| \leq 2^{\pi(X)}$.
(van Douwen, 1978)

If X is homogeneous then $|X| \leq |\mathrm{RO}(X)|^{\pi\chi(X)}$.
(Ismail, 1981)

If X is compact and power homogeneous then

(a) $|X| \leq w(X)^{\pi\chi(X)}$ (van Mill, 2004)
(b) $|X| \leq 2^{t(X)}$.
(Arhangelskii, van Mill, Ridderbos, 2005)

If X is Δ-power homogeneous then $|X| \leq d(x)^{\pi\chi(X)}$.
If in addition X is regular then $|X| \leq 2^{c(X)\pi\chi(X)}$.
(Ridderbos, 2005)

3. θ-density, semiregularization, and homogeneity.

A subspace $D \subseteq X$ is θ-**dense** if $D \cap \text{cl}_X U \neq \varnothing$ for every nonempty open set U of X. Let $d_\theta(X)$ denote the least cardinality of a θ-dense subset of X.

Given a point $x \in X$, a collection $\mathcal{B}$ of nonempty open sets is a **local** $\pi\theta$-**base at** x if for every open set U containing x there exists $B \in \mathcal{B}$ such that $\text{cl}_X B \subseteq \text{cl}_X U$. Let $\pi\chi_\theta(x, X)$, the $\pi\theta$-**character at** x, denote the least cardinality of a $\pi\theta$-base at x. Let $\pi\chi_\theta(X) = \sup_{x \in X} \pi\chi_\theta(x, X)$.

X is **semiregular** if RO(X) forms a basis for X. Let X_s, the **semiregularization** of X, denote the space with the same underlying set as X and RO(X) as a basis. X_s is necessarily semiregular and Hausdorff.

A space X is **quasiregular** if for every nonempty open set U there exists a non-empty open set V such that $\text{cl}_X V \subseteq U$.

Proposition. *If X is homogeneous, power homogeneous, or Δ-power homogeneous then X_s has the same property.*

Proposition. (a) $c(X_s) = c(X)$.

(b) $\pi\chi(X_s) = \pi\chi_\theta(X) \leq \pi\chi(X)$.

(c) $d_\theta(X) \leq d(X)$. *Equality occurs if X is quasiregular.*

Theorem. (C., 2005) *If X is Urysohn and X_s is Δ-power homogeneous then $|X| \leq d_\theta(X)^{\pi\chi_\theta(X)}$.*

Example. Let Y be the unit circle with its usual topology and let X be the unit circle with basis given by

$$\{U \backslash C : U \text{ is open in } Y \text{ and } C \in [X]^{<\mathfrak{c}}\}.$$

X is Urysohn and $X_s = Y$, a homogeneous space. (In fact X is homogeneous). The rationals, naturally projected onto the unit circle, are θ-dense in X. Hence $d_\theta(X) = \omega$. Also, it is easy to see that $\pi\chi_\theta(X) = \omega$, $d(X) = \mathfrak{c}$, and $\pi\chi(X) = \mathfrak{c}$. The Ridderbos bound gives

$$|X| \leq d(X)^{\pi\chi(X)} = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$$

while the bound given above gives

$$|X| \leq d_\theta(X)^{\pi\chi_\theta(X)} = \omega^\omega = \mathfrak{c}.$$

Theorem. (Šapirovskiĭ, 1974) *If X is regular then $d(X) \leq \pi\chi(X)^{c(X)}$.*

Theorem. *For any space X, $d_\theta(X) \leq \pi\chi(X)^{c(X)}$.*

Corollary. *If X is quasiregular then $d(X) \leq \pi\chi(X)^{c(X)}$.*

Theorem. (C., 2005) *Suppose X is Urysohn or quasiregular. If X_s is Δ-power homogenous then $|X| \leq 2^{c(X)\pi\chi_\theta(X)}$.*

Question. *If X is Hausdorff and X_s is Δ-power homogeneous, is it true that $|X| \leq d_\theta(X)^{\pi\chi_\theta(X)}$?*

Question. *If X is Hausdorff and X_s is Δ-power homogeneous, is it true that $|X| \leq 2^{c(X)\pi\chi_\theta(X)}$?*

5. A variation on a result of Kunen.

An **F-space** is a Tychonoff space in which every cozero set is C^*-embedded. Every extremely disconnected Tychonoff space is an F-space.

Theorem. (Kunen, 1990) *No compact F-space is power homogeneous.*

X is **H-closed** if it is closed in every Hausdorff space in which it is embedded, or equivalently, if every open cover of X has a finite subfamily whose union is dense in X.

Theorem. (C.,2005) *No H-closed extremely disconnected space is power homogeneous.*

Question. *Does there exist an H-closed power homogeneous space in which every cozero set is C^*-embedded?*