A new cardinality bound on homogeneous spaces via the Erdös-Rado Theorem

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Background

Assume all spaces are Hausdorff.

X is **homogeneous** if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

X is **power homogeneous** if there exists a cardinal κ such that X^κ is homogeneous.

Background, continued

- If X is power homogeneous, then $|X| \leq 2^{\pi w(X)}$. (van Douwen, 1978)
- Compare with $|X| \leq 2^{w(X)}$ for general spaces X.
- If X is power homogeneous, then $|X| \leq d(X)^{\pi\chi(X)}$. (Ridderbos, 2005)
- Compare with $|X| \leq d(X)^{\chi(X)}$ for general spaces X.
- $|X| \leq 2^{c(X)\chi(X)}$ for general spaces X. (Hajnal, Juhász, 1967)
- Question: If X is power homogeneous, is it true that $|X| \leq 2^{c(X)\pi\chi(X)}$? Denote this inequality by the Inquality.

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The Inequality is true if X is power homogeneous and

- compact (van Mill, 2004)
- regular (Ridderbos, 2005):
  Šapirovskiĭ showed that $d(X) \leq \pi\chi(X)^{c(X)}$ for regular spaces, so

$$|X| \leq d(X)^{\pi\chi(X)} \leq \left(\pi\chi(X)^{c(X)\pi\chi(X)}\right) = 2^{c(X)\pi\chi(X)}.$$

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The Erdös-Rado Theorem

Let X be set, κ a cardinal, and $f : X^2 \to \kappa$. If $|X| > 2^\kappa$ then there exists $\alpha < \kappa$ and $Y \in [X]^{\kappa^+}$ such that $f(x, y) = \alpha$ for all $\{x, y\} \in Y^2$. In other words,

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa .$$

Note:

- this is only one case of the Erdös-Rado Theorem.
- this theorem has been used to prove several cardinality bounds, notably $|X| \leq 2^{c(X)\chi(X)}$ for general spaces X.

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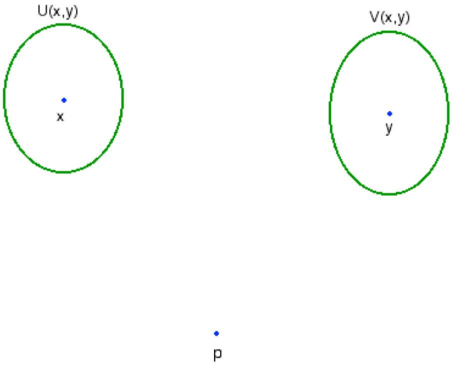
The Homogeneous Case

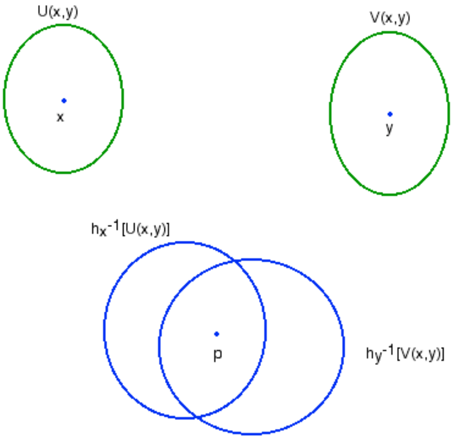
Theorem (C. and R., 2007)

If X is homogeneous, then $|X| \leq 2^{c(X)\pi\chi(X)}$.

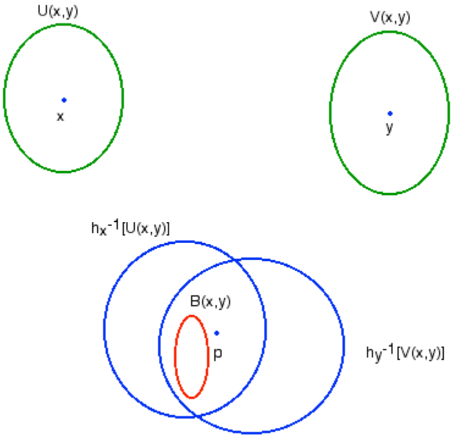
Fix $p \in X$ and a local π-base $\mathcal{B}$ at p such that $|B| \leq \pi\chi(X)$. For all $x \in X$, fix a homeo. $h_x : X \to X$ such that $h_x(p) = x$.



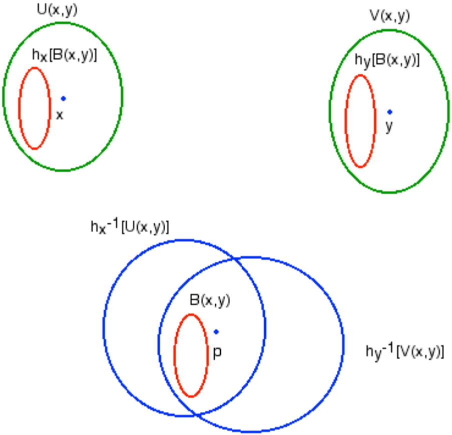
For all $x \neq y \in X$ there exist disjoint open sets $U(x, y)$ and $V(x, y)$ containing x and y respectively.



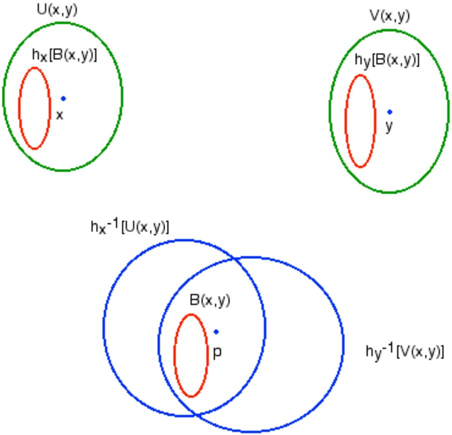
$p \in h_x^{\leftarrow}[U(x,y)] \cap h_y^{\leftarrow}[V(x,y)]$, an open set.



There exists $B(x,y) \in \mathcal{B}$ s.t.
$B(x,y) \subseteq h_x^{\leftarrow}[U(x,y)] \cap h_y^{\leftarrow}[V(x,y)]$.



$h_x[B(x,y)] \subseteq U(x,y)$ and $h_y[B(x,y)] \subseteq V(x,y)$.



Observe $h_x[B(x,y)] \cap h_y[B(x,y)] = \varnothing$.

- Let $\kappa = c(X)\pi\chi(X)$, and suppose by way of contradiction that $|X| > 2^\kappa$.
- Note that $B : X^2 \to \mathcal{B}$, and $|\mathcal{B}| \leq \kappa$. By the Erdös-Rado Theorem there exists $Y \in [X]^{\kappa^+}$ and $B \in \mathcal{B}$ such that $B(x, y) = B$ for all $\{x, y\} \in Y^2$.
- $\mathcal{C} = \{h_x[B] : x \in Y\}$ is a cellular family: for $x \neq y \in Y$, we have

$$h_x[B] \cap h_y[B] = h_x[B(x, y)] \cap h_y[B(x, y)] = \varnothing.$$

- But $|\mathcal{C}| = |Y| = \kappa^+ > \kappa \geq c(X)$, a contradiction. Hence

$$|X| \leq 2^\kappa.$$

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Remarks

- The Erdös-Rado Theorem has been used to prove several cardinality bounds, notably the Hajnal-Juhász result. However, the application of the Erdös-Rado Theorem to homogeneous spaces appears to be new.
- In previous proofs of the Inequality where the homogeneous space is either compact, regular, or Urysohn, $c(X)$ only came in as an afterthought via the result of Šapirovskiĭ for regular spaces, or a modified version for general spaces. However, in this proof the cellularity $c(X)$ is used directly in conjunction with the homogeneity.

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The Power Homogeneous Case

- Let X be a power homogeneous space, and let μ be a cardinal such that X^μ is homogeneous.
- Set $\Delta(X, \mu) = \{x \in X^\mu : \pi_\alpha(x) = \pi_\beta(x) \quad \forall \alpha, \beta \in \mu\}$.
- Fix $p \in \Delta(X, \mu)$.

Lemma (Very rough version)

For every $x \in \Delta(X, \mu)$ there is a homeomorphism $h_x : X^\mu \to X^\mu$ such that $h_x(p) = x$ and "h_x interacts well with projection maps and local π-bases at $\pi_A(p)$, where $A \subseteq \mu$".

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Theorem (C. and R., 2007)

If X is power homogeneous then $|X| \leq 2^{c(X)\pi\chi(X)}$.

Proof.

Ingredients:

- the previous Lemma.
- construction of a "large" cellular family in $X^{c(X)\pi\chi(X)}$, if $|X| > 2^{c(X)\pi\chi(X)}$.
- elements reminiscent of the "clustering" technique Van Douwens used to show $|X| \leq 2^{\pi w(X)}$ for power homogeneous spaces.
- the Erdös-Rado Theorem.

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The Homeomorphism Group $H(X)$

- For a space X, let $H(X)$ be the set of homeomorphisms $h : X \to X$.
- Frankiewicz showed in 1979 that $|H(X)| \leq 2^{\pi w(X)}$.
- A consequence is that $|X| \leq 2^{\pi w(X)}$ for homogeneous spaces X, since $|X| \leq |H(X)|$ for homogeneous spaces.

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Definition

A subset $Z \subseteq X$ **separates** $\mathcal{G} \subseteq H(X)$ if for all $f \neq g \in \mathcal{G}$ there is some $z \in Z$ such that $f(z) \neq g(z)$. The **separation degree** of X is defined by

$$sd(X) = \min\{|Z| : Z \text{ separates } H(X)\}.$$

Theorem (C. and R., 2007)

For a general space X, $|H(X)| \leq 2^{c(X)\pi\chi(X)sd(X)} \leq 2^{\pi w(X)}$.

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- There exists a c.c.c. Urysohn space X such that $d(X) = \omega_1$ and $sd(X)\pi\chi(X) \leq \omega$. Thus, under CH,

$$2^{c(X)\pi\chi(X)sd(X)} < 2^{\pi w(X)}.$$

- Assuming $\mathfrak{c}^+ = 2^{\mathfrak{c}}$, there exists a c.c.c Urysohn space such that

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Questions

1. In view of Arhangel'skii's inequality for general spaces, is it true that if X is power homogeneous then

   $$|X| \leq 2^{L(X)\pi\chi(X)}?$$

   At least is it true that if X is compact and power homogeneous then

   $$|X| \leq 2^{\pi\chi(X)}?$$

   Note that Arhangel'skii, van Mill, and Ridderbos have shown that if X is compact and power homogeneous then

   $$|X| \leq 2^{t(X)},$$

   and $\pi\chi(X) \leq t(X)$ for compact spaces.

2. Is it true that $|H(X)| \leq 2^{c(X)\pi\chi(X)}$ for general spaces X?